GROUP-LIKE ELEMENTS IN QUANTUM GROUPS AND FEIGIN'S CONJECTURE

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Abstract

Let A be an arbitrary symmetrizable Cartan matrix of rank r, and $\mathbf{n} = \mathbf{n}_+$ be the standard maximal nilpotent subalgebra in the Kac-Moody algebra associated with A (thus, $\mathbf{n}$ is generated by $E_1, \ldots, E_r$ subject to the Serre relations). Let $\hat{U}_q(\mathbf{n})$ be the completion (with respect to the natural grading) of the quantized enveloping algebra of $\mathbf{n}$. For a sequence $\mathbf{i} = (i_1, \ldots, i_m)$ with $1 \leq i_k \leq r$, let $P_\mathbf{i}$ be a skew polynomial algebra generated by $t_1, \ldots, t_m$ subject to the relations $t_l t_k = q^{C_{i_k, i_l}} t_k t_l$ $(1 \leq k < l \leq m)$ where $C = (C_{ij}) = (d_i a_{ij})$ is the symmetric matrix corresponding to A. We construct a group-like element $\mathbf{e_i} \in P_\mathbf{i} \bigotimes \hat{U}_q(\mathbf{n})$. This element gives rise to the evaluation homomorphism $\psi_\mathbf{i} : \mathbf{C}_q[N] \to P_\mathbf{i}$ given by $\psi_\mathbf{i}(x) = x(\mathbf{e_i})$, where $\mathbf{C}_q[N] = U_q(\mathbf{n})^0$ is the restricted dual of $U_q(\mathbf{n})$. Under a well-known isomorphism of algebras $\mathbf{C}_q[N]$ and $U_q(\mathbf{n})$, the map $\psi_\mathbf{i}$ identifies with Feigin's homomorphism $\Phi(\mathbf{i}) : U_q(\mathbf{n}) \to P_\mathbf{i}$. We prove that the image of $\psi_\mathbf{i}$ generates the skew-field of fractions $\mathcal{F}(P_\mathbf{i})$ if and only if $\mathbf{i}$ is a reduced expression of some element w in the Weyl group W; furthermore, in the latter case, Ker $\psi_\mathbf{i}$ depends only on w (so we denote $I_w :=$ Ker $\psi_\mathbf{i}$). This result generalizes the results in [5], [6] to the case of Kac-Moody algebras. We also construct an element $\mathcal{R}_w \in \left(\mathbf{C}_q[N]/I_w\right) \bigotimes \hat{U}_q(\mathbf{n})$ which specializes to $\mathbf{e_i}$ under the embedding $\mathbf{C}_q[N]/I_w \hookrightarrow P_\mathbf{i}$. The elements $\mathcal{R}_w$ are closely related to the quazi-R-matrix studied by G. Lusztig in [8]. If $\mathbf{i}, \mathbf{i}'$ are reduced expressions of the same element $w \in W$, we have a natural isomorphism $\mathbf{R}_\mathbf{i}^{\mathbf{i}'} : \mathcal{F}(P_\mathbf{i}) \to \mathcal{F}(P_{\mathbf{i}'})$ such that $(\mathbf{R}_\mathbf{i}^{\mathbf{i}'} \otimes \mathrm{id})(\mathbf{e_i}) = \mathbf{e}_{\mathbf{i}'}$. This leads to identities between quantum exponentials. The maps $\mathbf{R}_\mathbf{i}^{\mathbf{i}'}$ are q-deformations of Lusztig's transition maps [8]. The existence of the maps $\mathbf{R}_\mathbf{i}^{\mathbf{i}'}$ leads to a surprising combinatorial corollary about skew-symmetric matrices associated with reduced expressions (cf. [12]).

0. Introduction and main results

It is well-known that a quantum group is not a group. One of the goals of this chapter is to introduce group-like elements for quantum deformations of certain nilpotent algebraic groups. In this section, we sketch our main results; more details will be given in subsequent sections.

Consider a maximal unipotent subgroup N in a complex simple algebraic group G. The group-like elements will be obtained as quantum deformation of certain morphisms $\pi_{\mathbf{i}} : \mathbf{C}^m \to N$ defined as follows.

Let $E_1, E_2, \ldots, E_r$ be standard generators of $\mathbf{n}$, the Lie algebra of N. For any sequence $\mathbf{i} = (i_1, \ldots, i_m)$ of indices (possibly with repetitions), one defines a map $\mathbf{C}^m \to N$ by the formula

$$\pi_{\mathbf{i}}(t_1, \ldots, t_m) = \exp(t_1 E_{i_1})\exp(t_2 E_{i_2}) \cdots \exp(t_m E_{i_m}) \tag{0.1}$$

where $\exp : \mathbf{n} \to N$ is the exponential map. Note that $\pi_{\mathbf{i}}$ is a regular (algebraic) map.

It is well-known that $\pi_{\mathbf{i}}$ is a birational isomorphism $\mathbf{C}^m \cong N$ if $\mathbf{i} = (i_1, \ldots, i_m)$ is a reduced expression of w_0, the longest element in the Weyl group W of G. Furthermore, if $\mathbf{i}$ is a reduced expression of $w \in W$, then the closure in N of the image of $\pi_{\mathbf{i}}$ depends only on w.

To construct a q-deformation of $\pi_{\mathbf{i}}$ we interpret the evaluation homomorphism $\pi_{\mathbf{i}}^* : \mathbf{C}[N] \to \mathbf{C}[t_1, \ldots, t_m]$ as follows. First, we think of the product in (0.1) as an element

$$\tilde{\pi}_{\mathbf{i}} \in \mathbf{C}[t_1, \ldots, t_m] \bigotimes \hat{U}(\mathbf{n}),$$

where $\hat{U}(\mathbf{n})$ is the completion of the universal enveloping algebra of $\mathbf{n}$ with respect to the natural grading.

Second, $\mathbf{C}[N]$ can be identified with $U(\mathbf{n})^0$, the restricted dual Hopf algebra. This gives rise to a natural pairing $\mathbf{C}[N] \times \hat{U}(\mathbf{n}) \to \mathbf{C}$. Extending scalars from $\mathbf{C}$ to $P = \mathbf{C}[t_1, \ldots, t_m]$ we see that each $f \in \mathbf{C}[N]$ becomes a linear form on $P \bigotimes \hat{U}(\mathbf{n})$ with values in P. Then we have

$$\pi_{\mathbf{i}}^*(f) = f(\tilde{\pi}_{\mathbf{i}}) \ . \tag{0.2}$$

We construct the deformation of $\tilde{\pi}_{\mathbf{i}}$ in a more general situation when $\mathbf{n}$ is the standard maximal nilpotent Lie subalgebra in a Kac-Moody algebra $\mathbf{g}$. Let us briefly introduce necessary definitions and notation.

Let $A = (a_{ij})$ be a symmetrizable Cartan matrix of size $r \times r$. Denote by $C = (C_{ij}) = (d_i a_{ij})$ the corresponding symmetric matrix. Let $\mathcal{U}$ be the associative algebra over $\mathbf{C}(q)$ generated by $E_1, \ldots, E_r$ subject to the *quantum Serre relations* (this is the quantized universal enveloping algebra $U_q(\mathbf{n})$ of the nilpotent part of the Kac-Moody algebra corresponding to A). The algebra $\mathcal{U}$ is graded by $\mathbf{Z}_+^r$ via $\deg(E_i) = \alpha_i$, the

standard basis vector in $\mathbf{Z}_+^r$. Denote by $\hat{\mathcal{U}}$ the completion with respect to the grading. Following [8], Chapter 2, we consider $\mathcal{U}$ with the structure of a *braided bialgebra* with the *braided* coproduct $\Delta : \mathcal{U} \to \mathcal{U} \bigotimes \mathcal{U}$. Namely, $\Delta(E_i) = E_i \otimes 1 + 1 \otimes E_i$, and Δ is a homomorphism of $\mathbf{Z}_+^r$-graded algebras, where the algebra structure on the tensor square of $\mathcal{U}$ differs from the standard one by a twist (see Section 2 below for more details). It follows that $\hat{\mathcal{U}}$ is a *complete bialgebra* with the coproduct $\hat{\Delta} : \hat{\mathcal{U}} \to \hat{\mathcal{U}} \hat{\bigotimes} \hat{\mathcal{U}}$. The *quantum group* $\mathcal{A}$ is the restricted dual algebra of $\mathcal{U}$ (if A is of finite type then $\mathcal{A}$ can be identified with the q-deformed ring of polynomial functions $\mathbf{C}_q[N]$). The natural evaluation pairing $\mathcal{A} \times \hat{\mathcal{U}} \to \mathbf{C}(q)$ will be denoted by $(x, E) \mapsto x(E)$.

Let $\mathbf{i} = (i_1, \ldots, i_m)$ be a sequence of integers with $1 \le i_k \le r$. Denote by $P_\mathbf{i}$ the $\mathbf{C}(q)$-algebra generated by $t_1, \ldots, t_m$ subject to the relations

$$t_l t_k = q^{C_{i_k, i_l}} t_k t_l, \quad 1 \le k < l \le m \; . \tag{0.3}$$

Let $\hat{\mathcal{U}}_\mathbf{i} = P_\mathbf{i} \bigotimes \hat{\mathcal{U}}$ be the space of all series of the form

$$\sum_{\gamma \in \mathbf{Z}_+^r} t_\gamma \otimes E_\gamma$$

where $t_\gamma \in P_\mathbf{i}$ and $E_\gamma \in \mathcal{U}$ is a homogeneous element of degree γ. There is a standard algebra structure on $\hat{\mathcal{U}}_\mathbf{i}$. We identify $P_\mathbf{i} \bigotimes 1$ with $P_\mathbf{i}$ and $1 \bigotimes \hat{\mathcal{U}}$ with $\hat{\mathcal{U}}$ so that $tE = Et = t \otimes E$ in $\hat{\mathcal{U}}_\mathbf{i}$. Note also that $\hat{\mathcal{U}}_\mathbf{i}$ is a $P_\mathbf{i}$-bimodule in the standard way. The coproduct $\hat{\Delta}$ on $\hat{\mathcal{U}}$ extends naturally to the $P_\mathbf{i}$-bilinear map

$$\hat{\Delta}_\mathbf{i} : \hat{\mathcal{U}}_\mathbf{i} \to \hat{\mathcal{U}}_\mathbf{i} \hat{\bigotimes_{P_\mathbf{i}}} \hat{\mathcal{U}}_\mathbf{i}$$

by the formula: $\hat{\Delta}_\mathbf{i}(\sum_\gamma t_\gamma E_\gamma) = \sum_\gamma t_\gamma \Delta(E_\gamma)$. We call $\mathbf{e} \in \hat{\mathcal{U}}_\mathbf{i}$ a *group-like element* if $\hat{\Delta}_\mathbf{i}(\mathbf{e}) = \mathbf{e} \otimes \mathbf{e}$.

Finally, we define the q-exponential

$$\exp_q(u) = \sum_{n \ge 0} \frac{u^n}{[n]_q!} \tag{0.4}$$

where $[n]_q! = [1]_q [2]_q \cdots [n]_q$, $[l]_q = 1 + q + \cdots q^{(l-1)}$.

Now we can state our first main result.

Theorem 0.1. *For any sequence $\mathbf{i} = (i_1, \ldots, i_m)$ and any $c_1, \ldots, c_m \in \mathbf{C}(q)$, the product*

$$\exp_{q_{i_1}}(c_1 t_1 E_{i_1}) \exp_{q_{i_2}}(c_2 t_2 E_{i_2}) \cdots \exp_{q_{i_m}}(c_m t_m E_{i_m})$$

is a group-like *element in $\hat{\mathcal{U}}_{\mathbf{i}}$, where $q_i = q^{C_{ii}}$ for $i = 1, \ldots, r$.*

We prove Theorem 0.1 in Section 1 for more general *braided bialgebras*.

We denote

$$\mathbf{e_i} = \exp_{q_{i_1}}(t_1 E_{i_1}) \exp_{q_{i_2}}(t_2 E_{i_2}) \cdots \exp_{q_{i_m}}(t_m E_{i_m}) \ . \tag{0.5}$$

As in the commutative case, we extend the evaluation pairing $\mathcal{A} \times \hat{\mathcal{U}} \to \mathbf{C}(q)$ to the $P_{\mathbf{i}}$-linear pairing $\mathcal{A} \times \hat{\mathcal{U}}_{\mathbf{i}} \to P_{\mathbf{i}}$. As an analogue of (0.2) we define the map $\psi_{\mathbf{i}} : \mathcal{A} \to P_{\mathbf{i}}$ by

$$\psi_{\mathbf{i}}(x) := x(\mathbf{e_i}) \ . \tag{0.6}$$

Corollary 0.2. *The map $\psi_{\mathbf{i}}$ is an algebra homomorphism.*

Proof. The definition of the pairing $(x, E) \to x(E)$ implies that $(xy)(u) = (x \otimes y)(\hat{\Delta}(u))$ for all $x, y \in \mathcal{A}$ and $u \in \hat{\mathcal{U}}_{\mathbf{i}}$, where $(x \otimes y)(u_1 \otimes u_2) := x(u_1)y(u_2)$ for any $u_1, u_2 \in \hat{\mathcal{U}}_{\mathbf{i}}$. Thus, we have

$$\psi_{\mathbf{i}}(xy) = (xy)(\mathbf{e_i}) = (x \otimes y)(\hat{\Delta}_{\mathbf{i}}(\mathbf{e_i})) = (x \otimes y)(\mathbf{e_i} \otimes \mathbf{e_i}) = x(\mathbf{e_i})y(\mathbf{e_i}) = \psi_{\mathbf{i}}(x)\psi_{\mathbf{i}}(y) \ .$$

Corollary 0.2 is proved. $\triangleleft$

Expanding (0.5), we obtain the following formula for $\psi_{\mathbf{i}}$:

$$\psi_{\mathbf{i}}(x) = \sum_{a_1, a_2 \ldots, a_m \geq 0} x\big(E_{i_1}^{[a_1]} E_{i_2}^{[a_2]} \cdots E_{i_m}^{[a_m]}\big) t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m} \tag{0.7}$$

where $E_i^{[n]} = \dfrac{1}{[n]_{q_i}!} E_i^n$. Note that the sum in (0.7) is always finite.

Under a well-known isomorphism $\mathcal{A} \cong \mathcal{U}$ the homomorphism $\psi_{\mathbf{i}}$ becomes *Feigin's homomorphism* $\Phi(\mathbf{i}) : \mathcal{U} \to P_{\mathbf{i}}$ ([5] and Section 2 below).

Using the homomorphism $\psi_{\mathbf{i}}$, we can express the group-like element $\mathbf{e_i}$ in terms of the *universal element* $\mathcal{R} \in \mathcal{A} \bigotimes \hat{\mathcal{U}}$ defined as follows. Under the canonical isomorphism between $\mathcal{A} \bigotimes \hat{\mathcal{U}}$ and the space of linear maps $\mathcal{U} \to \hat{\mathcal{U}}$, the element $\mathcal{R}$ corresponds to the inclusion $\mathcal{U} \hookrightarrow \hat{\mathcal{U}}$.

Proposition 0.3. *For any sequence* $\mathbf{i}$ *as above, we have*

$$(\psi_{\mathbf{i}} \otimes \mathrm{id})(\mathcal{R}) = \mathbf{e_i} \ . \tag{0.8}$$

The element $\mathcal{R}$ is uniquely determined by the equations (0.8) for all $\mathbf{i}$.

Proof. Let B be a homogeneous basis in $\mathcal{U}$ (that is, B is compatible with the $\mathbf{Z}_+^r$-grading), and $\{b^*\}$ be the dual basis in $\mathcal{A}$ (so that $b^*(b') = \delta_{b,b'}$). By definition,

$$\mathcal{R} = \sum_{b \in B} b^* \otimes b \ . \tag{0.9}$$

We have

$$(\psi_{\mathbf{i}} \otimes \mathrm{id})(\mathcal{R}) = \sum_{b \in B} \psi_{\mathbf{i}}(b^*) \otimes b = \sum_{b \in B} b^*(\mathbf{e}_{\mathbf{i}}) \otimes b = \mathbf{e}_{\mathbf{i}}$$

by definition (0.6) of $\psi_{\mathbf{i}}$ and by the formula $\sum_{b \in B} b^*(u) \otimes b = u$ for any $u \in \hat{\mathcal{U}}_{\mathbf{i}}$.

It remains to check the uniqueness. Assume that there is another $\mathcal{R}'$ satisfying (0.8) for all $\mathbf{i}$. The equations (0.8) imply that $(\mathcal{R} - \mathcal{R}') \in \mathrm{Ker}\ \psi_{\mathbf{i}} \bigotimes \hat{\mathcal{U}}$. By (0.7), an element $x \in \mathcal{A}$ is in the kernel of each $\psi_{\mathbf{i}}$ if and only if x vanishes at all monomials in $E_1, \ldots, E_r$. Hence, $\mathcal{R}' = \mathcal{R}$, and we are done. $\triangleleft$

The element $\mathcal{R}$ was studied in [8], Chapter 4 in a slightly different setting; it is a version of the universal R-matrix for the "braided" quantum double of $\mathcal{U}$.

Let W be the Weyl group generated by simple reflections $s_1, \ldots, s_r : \mathbf{Z}^r \to \mathbf{Z}^r$ defined by $s_i(\alpha_j) = \alpha_i - a_{ij}\alpha_i$ for all i, j. We say that $\mathbf{i} = (i_1, \ldots, i_m)$ is a reduced expression of $w \in W$ if $w = s_{i_1} \cdots s_{i_m}$ and this factorization of w is the shortest possible. We denote by $R(w)$ the set of all reduced expressions of w. We also reserve notation w_0 for the longest element in W if W is finite.

For a sequence $\mathbf{i} = (i_1, \ldots, i_m)$ let $\mathcal{U}(\mathbf{i})$ be the subspace in $\mathcal{U}$ spanned by all monomials $E_{i_1}^{a_1} E_{i_2}^{a_2} \cdots E_{i_m}^{a_m}$. It is-well known ([7], Section 4.4, or [9]) that if $\mathbf{i} \in R(w)$ then $\mathcal{U}(\mathbf{i})$ depends only on w. So we denote $\mathcal{U}(w) := \mathcal{U}(\mathbf{i})$ for all $\mathbf{i} \in R(w)$. It is also well-known that $\mathcal{U}(w_0) = \mathcal{U}$ if W is finite. Further, if $\mathbf{i}$ is not reduced then there is a subsequence $\mathbf{i}'$ of $\mathbf{i}$ such that $\mathbf{i}'$ is a reduced expression and $\mathcal{U}(\mathbf{i}) = \mathcal{U}(\mathbf{i}')$. For the convenience of the reader we will prove these assertions in Section 2.

Now we can give a complete description of $\mathrm{Ker}\ \psi_{\mathbf{i}}$ using (0.7) and the above discussion.

Lemma 0.4. *The kernel of $\psi_{\mathbf{i}}$ is the orthogonal complement of $\mathcal{U}(\mathbf{i})$:*

$$\mathrm{Ker}\ \psi_{\mathbf{i}} = \{x \in \mathcal{A} : x(u) = 0 \text{ for all } u \in \mathcal{U}(\mathbf{i})\} \ .$$

Furthermore,

(i) If $\mathbf{i}$ is not reduced then there is a reduced subsequence $\mathbf{i}'$ of $\mathbf{i}$ such that $\mathrm{Ker}\ \psi_{\mathbf{i}} = \mathrm{Ker}\ \psi_{\mathbf{i}'}$.

(ii) For every $w \in W$ and $\mathbf{i}, \mathbf{i}' \in R(w)$ we have $\mathrm{Ker}\ \psi_{\mathbf{i}} = \mathrm{Ker}\ \psi_{\mathbf{i}'}$.

(iii) If W is finite then $\mathrm{Ker}\ \psi_{\mathbf{i}} = \{0\}$ for any $\mathbf{i} \in R(w_0)$.

Denote $I_w := \mathrm{Ker}\ \psi_{\mathbf{i}}$ for $\mathbf{i} \in R(w)$. Our next main result is the following.

Theorem 0.5. *For every $w \in W$ and $\mathbf{i} \in R(w)$, the image of $\psi_{\mathbf{i}}$ generates the (skew) field of fractions of $P_{\mathbf{i}}$. Hence, $\psi_{\mathbf{i}}$ induces an isomorphism of the fields of fractions*

$$\overline{\psi}_{\mathbf{i}} : \mathcal{F}(\mathcal{A}/I_w) \tilde{\to} \mathcal{F}(P_{\mathbf{i}}) \ . \tag{0.10}$$

In particular, if W is finite and $w = w_0$ then $\overline{\psi}_{\mathbf{i}}$ is an isomorphism between $\mathcal{F}(\mathcal{A})$ and $\mathcal{F}(P_{\mathbf{i}})$.

Here the symbol $\mathcal{F}$ stands for the skew-field of fractions. We review the necessary definitions and results in the appendix below.

The last statement in Theorem 0.5 coincides with Feigin's conjecture ([5], [6]) (stated for $\mathcal{U}$ rather than for $\mathcal{A}$). The conjecture was proved in [5] for the type A_r and $w = w_0$ by a direct computation involving some specific reduced word $\mathbf{i} \in R(w_0)$. The conjecture was further generalized by A. Joseph ([6]) to any $w \in W$ in the assumption that W is finite, and proved by geometric arguments.

We give an algebraic proof of Theorem 0.5 in Section 2 without the assumption that W is finite. The following proposition plays the crucial role in the proof.

Proposition 0.6. *For every reduced expression* $\mathbf{i}$ *of some* $w \in W$*, there is an element* $x = x(\mathbf{i}) \in \mathcal{A}$ *such that* $\psi_{\mathbf{i}}(x) = t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m}$ *with* $a_1 > 0$.

We prove Proposition 0.6 in Section 3.

Corollary 0.7. *The skew-field* $\mathcal{F}\big(\psi_{\mathbf{i}}(\mathcal{A})\big)$ *coincides with* $\mathcal{F}(P_{\mathbf{i}})$ *if and only if* $\mathbf{i}$ *is a reduced expression.*

Proof. We need to prove the "only if" part (the "if" part is the assertion of Theorem 0.5). Assume that $\mathbf{i} = (i_1, \ldots, i_m)$ is not reduced. Then, by Lemma 0.4(i) and Theorem 0.5, there is a reduced subsequence $\mathbf{i}' = (i_{k_1}, \ldots, i_{k_l})$ of $\mathbf{i}$ such that Im $\psi_{\mathbf{i}'} \cong$ Im $\psi_{\mathbf{i}}$. It follows that $\mathcal{F}(\text{Im } \psi_{\mathbf{i}}) \cong \mathcal{F}(\text{Im } \psi_{\mathbf{i}'}) \cong \mathcal{F}(P_{\mathbf{i}'})$. But $\mathcal{F}(P_{\mathbf{i}'}) \not\cong \mathcal{F}(P_{\mathbf{i}})$ since these skew-fields have different Gel'fand-Kirillov dimensions (see [12], Proposition 2.18). Corollary 0.7 is proved. $\triangleleft$

Let us illustrate the above results in the case when A is of type A_{n-1}. In this case, one can show that $\mathcal{A}$ is generated by the elements x_{ij} with $1 \le i < j \le n$ subject to the following relations (cf. [2], [4]):

$$x_{ij}x_{kl} = x_{kl}x_{ij}, \ x_{il}x_{jk} = x_{jk}x_{il} \quad (1 \le i < j < k < l \le n) \ , \tag{0.11}$$

$$x_{ik} = \frac{qx_{ij}x_{jk} - x_{jk}x_{ij}}{q - q^{-1}}, \quad x_{ij}x_{ik} = qx_{ik}x_{ij}, \quad x_{jk}x_{ik} = q^{-1}x_{ik}x_{jk} \ (1 \le i < j < k \le n) \ .$$

The elements x_{ij} are q-deformations of the matrix entries considered as polynomial functions on the group N of the unipotent upper-triangular matrices. So we arrange the x_{ij} into the matrix $X = I + \sum\limits_{i<j} x_{ij}E_{ij}$, where I is the identity matrix, and the E_{ij} are the matrix units. Let $\psi_{\mathbf{i}}(X)$ be the $n \times n$-matrix (over $P_{\mathbf{i}}$) obtained from X by applying $\psi_{\mathbf{i}}$ to each matrix entry. The following proposition will be proved in Section 2.

Proposition 0.8. *For any sequence* $\mathbf{i} = (i_1, \ldots, i_m)$*, the matrix* $\psi_{\mathbf{i}}(X)$ *admits the following matrix factorization*

$$\psi_{\mathbf{i}}(X) = (I + t_1E_{i_1,i_1+1})(I + t_2E_{i_2,i_2+1}) \cdots (I + t_mE_{i_m,i_m+1}) \ . \tag{0.12}$$

If $\mathbf{i} \in R(w_0)$ then, applying the inverse isomorphism $(\overline{\psi}_{\mathbf{i}})^{-1} : \mathcal{F}(P_{\mathbf{i}}) \to \mathcal{F}(\mathcal{A})$ to the factorization (0.12), we obtain the factorization of the matrix X over $\mathcal{F}(\mathcal{A})$:

$$X = (I + \tilde{t}_1 E_{i_1,i_1+1})(I + \tilde{t}_2 E_{i_2,i_2+1}) \cdots (I + \tilde{t}_m E_{i_m,i_m+1}) \ ,$$

where $\tilde{t}_k = (\overline{\psi}_{\mathbf{i}})^{-1}(t_k)$. This factorization is a q-deformation of the one studied in [1]; it can be shown that such a factorization is unique. The explicit formulas for $\tilde{t}_k$ in terms of the matrix entries x_{ij} will be given in a separate publication. The above factorizations of the matrix X (and, more generally, $\mathcal{R}$ for quantum groups of finite type) were studied in [11].

Let us return to the general situation and discuss some corollaries of Theorem 0.5. For every $w \in W$ and $\mathbf{i}, \mathbf{i}' \in R(w)$ there is an isomorphism of skew fields

$$\mathbf{R}_{\mathbf{i}}^{\mathbf{i}'} : \mathcal{F}(P_{\mathbf{i}}) \widetilde{\to} \mathcal{F}(P_{\mathbf{i}'})$$

defined by $\mathbf{R}_{\mathbf{i}}^{\mathbf{i}'} = \overline{\psi}_{\mathbf{i}'} \circ (\overline{\psi}_{\mathbf{i}})^{-1}$. We extend it to the isomorphism $\mathbf{R}_{\mathbf{i}}^{\mathbf{i}'} \otimes \mathrm{id} : \mathcal{F}(P_{\mathbf{i}}) \bigotimes \hat{\mathcal{U}} \to \mathcal{F}(P_{\mathbf{i}'}) \bigotimes \hat{\mathcal{U}}$. In the following proposition, every element $\mathbf{e}_{\mathbf{i}}$ given by (0.5) is regarded as an element of $\mathcal{F}(P_{\mathbf{i}}) \bigotimes \hat{\mathcal{U}}$.

Proposition 0.9. *For every $\mathbf{i}, \mathbf{i}' \in R(w)$, we have $(\mathbf{R}_{\mathbf{i}}^{\mathbf{i}'} \otimes \mathrm{id})(\mathbf{e}_{\mathbf{i}}) = \mathbf{e}_{\mathbf{i}'}$.*

Proof. Let $\mathbf{p}_w : \mathcal{A} \to \mathcal{A}/I_w$ be the canonical projection. Denote $\mathcal{R}_w = (\mathbf{p}_w \otimes \mathrm{id})(\mathcal{R})$. Note that, similarly to $\mathcal{R}$, the element $\mathcal{R}_w \in \mathcal{A}/I_w \bigotimes \hat{\mathcal{U}}$ corresponds to the inclusion $\mathcal{U}(w) \hookrightarrow \mathcal{U}$. Proposition 0.3 implies that

$$(\overline{\psi}_{\mathbf{i}} \otimes \mathrm{id})(\mathcal{R}_w) = \mathbf{e}_{\mathbf{i}} \tag{0.13}$$

for every $\mathbf{i} \in R(w)$. We are done since $\mathbf{R}_{\mathbf{i}}^{\mathbf{i}'} \otimes \mathrm{id} = (\overline{\psi}_{\mathbf{i}'} \otimes \mathrm{id}) \circ (\overline{\psi}_{\mathbf{i}} \otimes \mathrm{id})^{-1}$. $\triangleleft$

Proposition 0.9 implies some identities between quantum exponentials. For two reduced expressions $\mathbf{i} = (i_1, \ldots, i_m)$ and $\mathbf{i}' = (i'_1, \ldots, i'_m)$ of an element $w \in W$, let $t_1, \ldots, t_m$ (resp. $t'_1, \ldots, t'_m$) be the standard generators of $P_{\mathbf{i}}$ (resp. $P_{\mathbf{i}'}$).

Corollary 0.10. *The following identity holds in the algebra $\mathcal{F}(P_{\mathbf{i}}) \bigotimes \hat{\mathcal{U}}$:*

$$\exp_{q_{i_1}}(t_1 E_{i_1}) \cdots \exp_{q_{i_m}}(t_m E_{i_m}) = \exp_{q_{i'_1}}(p_1 E_{i'_1}) \cdots \exp_{q_{i'_m}}(p_m E_{i'_m}) \ , \tag{0.14}$$

where $p_k = \mathbf{R}_{\mathbf{i}'}^{\mathbf{i}}(t'_k)$ for $k = 1, \ldots, m$. Identity (0.14) remains true under the rescaling $E_i \mapsto c_i E_i$ for any $c_i \in \mathbf{C}(q)$, $i = 1, \ldots, r$.

Example 1. Let A be the Cartan matrix of type A_2 . Then the Weyl group W is the symmetric group S_3, and $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$. Take $\mathbf{i} = (121)$, $\mathbf{i}' = (212)$, and denote by t_1, t_2, t_3 and t'_1, t'_2, t'_3 the generators of $P_{(121)}$ and $P_{(212)}$ respectively. Then $\mathbf{R}_{(212)}^{(121)}(t'_k) = p_k$ for $k = 1, 2, 3$, where

$$p_1 = t_2 t_3 (t_1 + t_3)^{-1}, \ p_2 = t_1 + t_3, \ p_3 = (t_1 + t_3)^{-1} t_1 t_2 \ .$$

This is a consequence of the matrix equation

$$(I+p_1E_{23})(I+p_2E_{12})(I+p_3E_{23}) = (I+t_1E_{12})(I+t_2E_{23})(I+t_3E_{12}) \ ,$$

which follows from the factorization (0.12).

The identity (0.14) takes the form

$$\exp_{q^2}(c_1t_1E_1)\exp_{q^2}(c_2t_2E_2)\exp_{q^2}(c_1t_3E_1) = \exp_{q^2}(c_2p_1E_2)\exp_{q^2}(c_1p_2E_1)\exp_{q^2}(c_2p_3E_2) \tag{0.15}$$

for any $c_1, c_2 \in \mathbf{C}(q)$. Expanding both sides of (0.15) and comparing the components of degree $2\alpha_1 + \alpha_2$, we obtain the quantum Serre relation

$$E_1^2E_2 - (q+q^{-1})E_1E_2E_1 - E_2E_1^2 = 0 \ .$$

We also note that setting $c_1 = 1$ and $c_2 = 0$ in (0.15) yields the familiar rule

$$\exp_{q^2}(t_1E_1)\exp_{q^2}(t_3E_1) = \exp_{q^2}((t_1+t_3)E_1) \ .$$

The identity (0.15) appeared in [11], Section 10.4; it was proved there by a straightforward computation.

We conclude the introduction by a surprising combinatorial consequence of the above results. To a sequence $\mathbf{i} = (i_1, \ldots, i_m)$ we associate a skew-symmetric $m \times m$-matrix $S(\mathbf{i})$ by the formula:

$$S(\mathbf{i}) = \sum_{1 \le k < l \le m} C_{i_k, i_l}(E_{kl} - E_{lk}) \ . \tag{0.16}$$

We say that two $m \times m$ matrices S and S' are *equivalent* if there is a matrix $T \in SL_m(\mathbf{Z})$ such that $S' = TST^t$ (where T^t is the transpose of T).

Proposition 0.11. *For every w and $\mathbf{i}, \mathbf{i}' \in R(w)$, the matrices $S(\mathbf{i})$ and $S(\mathbf{i}')$ are equivalent.*

This follows from the fact that the skew-fields $\mathcal{F}(P_{\mathbf{i}})$ and $\mathcal{F}(P_{\mathbf{i}'})$ are isomorphic, in view of a general result by A. Panov [12] (see also Section 2 below). Proposition 0.11 essentially says that there exists an isomorphism $\mathcal{F}(P_{\mathbf{i}'}) \to \mathcal{F}(P_{\mathbf{i}})$ which takes every generator t'_k to a monomial in $t_1, \ldots, t_m$. (Note that the isomorphism $\mathbf{R}_{\mathbf{i}'}^{\mathbf{i}}$ considered above, in general does not have this property).

The material is organized as follows. In Section 1 we introduce braided bialgebras and prove some results about them, including the generalization of Theorem 0.1. The quantum group $\mathcal{A}$ associated with a symmetrizable Cartan matrix is studied in Section 2, which contains the proofs of Lemma 0.4, Theorem 0.5 (modulo Proposition 0.6), and

Propositions 0.8 and 0.11. Section 3 is devoted to the proof of Proposition 0.6; our proof is based on the properties of extremal vectors in simple $U_q(\mathbf{g})$-modules. In Appendix we review necessary definitions and results about non-commutative fields of fractions.

Acknowledgments. The work presents results first appeared in my Ph.D. dissertation (Northeastern University, May 1996). I am very grateful to my advisor Professor Andrei Zelevinsky, for the continual support and help throughout the writing of this paper. This work benefited greatly from my conversations with J. Bernstein, G. Lusztig, S. Majid, A. Postnikov, Y. Soibelman, and J. Towber. I am especially indebted to Professor David Kazhdan of Harvard, first for making me welcome at Harvard, and second for teaching me geometric and categorical approach to the subject. Thanks are due the faculty of Department of Mathematics at Northeastern University, for the help during my stay there.

1. Results on braided bialgebras

Let $\mathbf{k}$ be a field and $\mathcal{U}$ be a $\mathbf{Z}_+^r$-graded $\mathbf{k}$-algebra: $\mathcal{U} = \bigoplus \mathcal{U}(\gamma)$, the sum over $\gamma \in \mathbf{Z}_+^r$. We assume that $\mathcal{U}(0) = \mathbf{k}$ and every $\mathcal{U}(\gamma)$ is finite-dimensional. Let $\mathbf{q} = (q_{ij})$, $1 \le i, j \le r$ be a $r \times r$-matrix with all $q_{ij} \in \mathbf{k}, q_{ij} \neq 0$. Following G. Lusztig ([8]) we associate with $\mathbf{q}$ an algebra structure on the vector space $\mathcal{U} \bigotimes \mathcal{U}$. For any two homogeneous elements $b \in \mathcal{U}(m_1, \ldots, m_r)$ and $c \in \mathcal{U}(n_1, \ldots, n_r)$ we set

$$Q(b, c) = Q((m_1, \ldots, m_r), (n_1, \ldots, n_r)) = \prod_{i,j=1}^{r} q_{ij}^{m_i n_j} \ . \tag{1.1}$$

We define the $\mathbf{q}$-*braided* multiplication in $\mathcal{U} \bigotimes \mathcal{U}$ by

$$(a \otimes b)(c \otimes d) := Q(b, c)(ac \otimes bd) \tag{1.2}$$

for any homogeneous elements b, c of $\mathcal{U}$ and any $a, d \in \mathcal{U}$.

It is easy to see that (1.2) makes $\mathcal{U} \bigotimes \mathcal{U}$ into a $\mathbf{Z}_+^r$-graded associative algebra (with the standard grading $(\mathcal{U} \bigotimes_{\mathbf{k}} \mathcal{U})(\gamma) = \bigoplus_{\gamma'} \ \mathcal{U}(\gamma') \bigotimes \mathcal{U}(\gamma - \gamma'))$. This algebra will be denoted by $\mathcal{U} \bigotimes_{\mathbf{q}} \mathcal{U}$ and called the $\mathbf{q}$-*braided* tensor square of $\mathcal{U}$.

We call $\mathcal{U}$ a $\mathbf{q}$-*braided bialgebra* if

(i) there is a homomorphism of $\mathbf{Z}_+^r$-graded algebras $\Delta : \mathcal{U} \to \mathcal{U} \bigotimes_{\mathbf{q}} \mathcal{U}$ satisfying the coassociativity constrain (we call Δ the *coproduct*);

(ii) There is a *counit* homomorphism of algebras $\varepsilon : \mathcal{U} \to \mathcal{U}(0) = \mathbf{k}$ satisfying

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}, \ \varepsilon(1) = 1. \tag{1.3}$$

This definition implies that for every $u \in \mathcal{U}$,

$$\Delta(u) = u \otimes 1 + 1 \otimes u + \sum_n u_n \otimes u_n' \tag{1.4}$$

where all u_n, u_n' are homogeneous elements of nonzero degrees. In particular, $\Delta(u) = u\otimes 1 + 1\otimes u$ for every $x \in \mathcal{U}(\alpha_1)\bigoplus\cdots\bigoplus\mathcal{U}(\alpha_r)$ where $\alpha_1,\ldots,\alpha_r$ is the standard basis in $\mathbf{Z}_+^r$. Another consequence of this definition is that $\varepsilon(x) = 0$ for any $x\in\mathcal{U}(\gamma), \gamma\neq 0$.

Note that the algebra $\mathcal{U}$ from the introduction, associated to a symmetrizable Cartan matrix A, is a $\mathbf{q}$-braided algebra, where $q_{ij} = q^{C_{ij}}$. Another example is the *free* algebra generated by $E_1,\ldots,E_r$, where $\mathbf{q}$ is arbitrary.

Let $\hat{\mathcal{U}}$ be the completion of $\mathcal{U}$ with respect to the grading, that is, the space of all formal series $\hat{u} = \sum\limits_{\gamma\in\mathbf{Z}_+^r} u_\gamma$, where $u_\gamma\in\mathcal{U}(\gamma)$. Clearly, $\hat{\mathcal{U}}$ is an algebra. The coproduct in $\mathcal{U}$ extends to $\hat\Delta : \hat{\mathcal{U}} \to \hat{\mathcal{U}}\hat{\bigotimes}_{\mathbf{q}}\hat{\mathcal{U}}$ so $\hat{\mathcal{U}}$ becomes a *complete bialgebra.*

Now we fix a positive integer m and consider a sequence $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ of integers with $1\le i_k\le r$. Let $\mathbf{q} = (q_{ij})$ be the matrix used in the definition of $\mathcal{U}$. Consider a $\mathbf{k}$-algebra $P_{\mathbf{i}} = P_{\mathbf{i},\mathbf{q}}$ generated by $t_1,\ldots,t_m$ subject to the following relations:

$$t_l t_k = q_{i_k,i_l} t_k t_l \tag{1.5}$$

for all $1\le k<l\le m$.

Define $\hat{\mathcal{U}}_{\mathbf{i}} = P_{\mathbf{i}}\bigotimes_{\mathbf{k}}\hat{\mathcal{U}}$, the space of formal series of the form $\sum_\gamma t_\gamma\otimes u_\gamma$, where $t_\gamma\in P_{\mathbf{i}}$ and $u_\gamma\in\mathcal{U}(\gamma)$. We consider $\hat{\mathcal{U}}_{\mathbf{i}}$ with the standard algebra structure (so we can write $tu = ut = t\otimes u$).

Consider the completed tensor square $\mathcal{V}_{\mathbf{i}} = \hat{\mathcal{U}}_{\mathbf{i}}\hat{\bigotimes\limits_{P_{\mathbf{i}}}}\hat{\mathcal{U}}_{\mathbf{i}}$ where the left factor is regarded as a right $P_{\mathbf{i}}$-module and the right factor as a left $P_{\mathbf{i}}$-module. Note that $\mathcal{V}_{\mathbf{i}}$ is a $P_{\mathbf{i}}$-bimodule. In $\mathcal{V}_{\mathbf{i}}$, we can write $t(u\otimes v) = (tu)\otimes v = u\otimes(tv) = (u\otimes v)t$ for any $u,v\in\mathcal{U}, t\in P_{\mathbf{i}}$. Under the standard identification $\mathcal{V}_{\mathbf{i}}\cong P_{\mathbf{i}}\bigotimes\hat{\mathcal{U}}\hat{\bigotimes}_{\mathbf{q}}\hat{\mathcal{U}}$ this bimodule $\mathcal{V}_{\mathbf{i}}$ becomes an algebra.

There is a natural morphism of $P_{\mathbf{i}}$-bimodules

$$\hat\Delta_{\mathbf{i}} : \hat{\mathcal{U}}_{\mathbf{i}} \to \mathcal{V}_{\mathbf{i}}$$

which is the $P_{\mathbf{i}}$-linear extension of the coproduct $\hat\Delta$ on $\hat{\mathcal{U}}$. Clearly, $\hat\Delta_{\mathbf{i}}$ is an algebra homomorphism.

Let $\mathbf{E} = (E_1,\ldots,E_m)$ be the family of elements $E_k\in\mathcal{U}(\alpha_{i_k})$. We define an element $\mathbf{e}_{\mathbf{i}} = \mathbf{e}_{\mathbf{i},\mathbf{E}}\in\hat{\mathcal{U}}_{\mathbf{i}}$ as follows:

$$\mathbf{e}_{\mathbf{i},\mathbf{E}} = \exp_{q_1}(t_1E_1)\exp_{q_2}(t_2E_2)\cdots\exp_{q_m}(t_mE_m) \tag{1.6}$$

where $q_k = q_{i_k,i_k}$ for $k = 1,\ldots,m$, and $\exp_{q_k}$ stands for the quantum exponential defined by (0.4).

The following result extends Theorem 0.1 to arbitrary $\mathbf{q}$-braided algebras.

Theorem 1.1. *For any sequence* $\mathbf{i}$ *and any family* $\mathbf{E} = (E_k)$ *as above the element* $\mathbf{e}_{\mathbf{i}} = \mathbf{e}_{\mathbf{i},\mathbf{E}}$ *is a* group-like *element in* $\hat{\mathcal{U}}_{\mathbf{i}}$*, i.e.,* $\hat\Delta_{\mathbf{i}}(\mathbf{e}_{\mathbf{i}}) = \mathbf{e}_{\mathbf{i}}\otimes\mathbf{e}_{\mathbf{i}}$.

Proof . We need the following.

Lemma 1.2.
(a) Each factor $\mathbf{e}_k = \exp_{q_k}(t_k E_k)$ *of* $\mathbf{e_i}$ *is a group-like element in* $\hat{\mathcal{U}}_{\mathbf{i}}$.
(b) $(1 \otimes \mathbf{e}_k)(\mathbf{e}_l \otimes 1) = (\mathbf{e}_l \otimes 1)(1 \otimes \mathbf{e}_k)$ *for any* $1 \le k < l \le m$.

Proof. (a) Denote $E = t_k E_k$. Since $\Delta(E_k) = E_k \otimes 1 + 1 \otimes E_k$, for each k we have

$$\hat{\Delta}_{\mathbf{i}}(E) = t_k(E_k \otimes 1 + 1 \otimes E_k) = E \otimes 1 + 1 \otimes E.$$

Denote $x = E \otimes 1, y = 1 \otimes E$. Let us show that that $yx = qxy$ where $q := q_k = q_{i_k,i_k}$. Indeed,

$$yx = (1 \otimes E)(E \otimes 1) = (1 \otimes t_k E_k)(t_k E_k \otimes 1) = t_k^2(1 \otimes E_k)(E_k \otimes 1)$$

$$= Q(E_k, E_k)t_k^2(E_k \otimes E_k) = q_{i_k,i_k}(t_k E_k \otimes t_k E_k) = qxy \ .$$

Further, we obtain

$$\hat{\Delta}_{\mathbf{i}}(\exp_q(E)) = \exp_q(\hat{\Delta}_{\mathbf{i}}(E)) = \exp_q(x + y)$$

and

$$\exp_q(E) \otimes \exp_q(E) = (\exp_q(E) \otimes 1)(1 \otimes \exp_q(E))$$

$$= (\exp_q(E \otimes 1))(\exp_q(1 \otimes E)) = \exp_q(x)\exp_q(y) \ .$$

Then the well-known rule for the quantum exponentials.

$$\exp_q(x + y) = \exp_q(x)\exp_q(y)$$

(provided that $yx = qxy$) implies that $\hat{\Delta}_{\mathbf{i}}(\exp_q(E)) = \exp_q(E) \otimes \exp_q(E)$. Part (a) is proved.

(b) Denote $E = t_k E_k$ and $E' = t_l E_l$. By definition of $\mathcal{U} \bigotimes_{\mathbf{q}} \mathcal{U}$,

$$(1 \otimes E)(E' \otimes 1) = t_k t_l (1 \otimes E_k)(E_l \otimes 1) = q_{i_k,i_l} t_k t_l (E_l \otimes E_k) \ .$$

The commutation relations (1.5) imply that

$$(1 \otimes E)(E' \otimes 1) = t_k t_l (1 \otimes E_k)(E_l \otimes 1) = t_l t_k (E_l \otimes E_k) = E' \otimes E = (E' \otimes 1)(1 \otimes E).$$

It follows that $(1 \otimes f(E))(g(E') \otimes 1) = f(E') \otimes g(E) = (f(E') \otimes 1)(1 \otimes g(E))$ for any polynomials f and g in one variable. Passing to the completion, we see that f and g can also be power series in the above formula. Taking $f(E) := \mathbf{e}_k = \exp_{q_k}(E)$ and $g(E') := \mathbf{e}_l = \exp_{q_l}(E')$ completes the proof of part (b). Lemma 1.2 is proved. $\triangleleft$

We are ready to complete the proof of Theorem 1.1 now. Recall that we use the shorthand $\mathbf{e}_k = \exp_{q_k}(t_k E_k)$ so $\mathbf{e_i} = \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_m$.

Using Lemma 1.2 and the fact that $(a \otimes 1)(1 \otimes b) = a \otimes b$ for any $a, b \in \hat{\mathcal{U}}_{\mathbf{i}}$, we obtain

$$\hat{\Delta}_{\mathbf{i}}(\mathbf{e}_{\mathbf{i}}) = \hat{\Delta}_{\mathbf{i}}(\mathbf{e}_1\mathbf{e}_2 \cdots \mathbf{e}_m) = \hat{\Delta}_{\mathbf{i}}(\mathbf{e}_1)\hat{\Delta}_{\mathbf{i}}(\mathbf{e}_2) \cdots \hat{\Delta}_{\mathbf{i}}(\mathbf{e}_m) = (\mathbf{e}_1 \otimes \mathbf{e}_1)(\mathbf{e}_2 \otimes \mathbf{e}_2) \cdots (\mathbf{e}_m \otimes \mathbf{e}_m)$$

$$= (\mathbf{e}_1 \otimes 1)(1 \otimes \mathbf{e}_1)(\mathbf{e}_2 \otimes 1)(1 \otimes \mathbf{e}_2) \cdots (\mathbf{e}_m \otimes 1)(1 \otimes \mathbf{e}_m) \ .$$

Using the commutativity property in Lemma 1.2(b), we obtain

$$\hat{\Delta}_{\mathbf{i}}(\mathbf{e}_{\mathbf{i}}) = \big((\mathbf{e}_1 \otimes 1)(\mathbf{e}_2 \otimes 1) \cdots (\mathbf{e}_m \otimes 1)\big)\big((1 \otimes \mathbf{e}_1)(1 \otimes \mathbf{e}_2) \cdots (1 \otimes \mathbf{e}_m)\big)$$

Finally, using the identities $(u \otimes 1)(v \otimes 1) = uv \otimes 1, (1 \otimes u)(1 \otimes v) = 1 \otimes uv$ for any $u, v \in \hat{\mathcal{U}}$, we obtain $\hat{\Delta}_{\mathbf{i}}(\mathbf{e}_{\mathbf{i}}) = (\mathbf{e}_{\mathbf{i}} \otimes 1)(1 \otimes \mathbf{e}_{\mathbf{i}}) = \mathbf{e}_{\mathbf{i}} \otimes \mathbf{e}_{\mathbf{i}}$. Theorem 1.1 is proved. $\triangleleft$

Now we define the *restricted dual* algebra $\mathcal{A} = \mathcal{U}^0$ of $\mathcal{U}$. As a vector space, $\mathcal{A}$ is the set of all $\mathbf{k}$-linear forms $x : \mathcal{U} \to \mathbf{k}$ such that x vanishes on $\mathcal{U}(\gamma)$ for all but finitely many $\gamma \in \mathbf{Z}_+^r$. In other words, $\mathcal{A} \cong \bigoplus_\gamma \mathcal{A}(\gamma)$ where $\mathcal{A}(\gamma) = \mathrm{Hom}_{\mathbf{k}}(\mathcal{U}(\gamma), \mathbf{k})$.

We define the multiplication $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ by the formula $(xy)(u) = (x \otimes y)\big(\Delta(u)\big)$ where $(x \otimes y)\big(u_1 \otimes u_2\big) = x(u_1)y(u_2)$. Thus, $\mathcal{A}$ becomes a $\mathbf{Z}_+^r$-graded algebra (with the unit $\mathbf{k} \to \mathcal{A}$ dual to the counit $\varepsilon : \mathcal{U} \to \mathbf{k}$).

Denote by $(x, u) \mapsto x(u)$ the natural non-degenerate evaluation pairing $\mathcal{A} \times \mathcal{U} \to \mathbf{k}$. Furthermore, we define the pairing $\mathcal{A} \times \hat{\mathcal{U}}_{\mathbf{i}} \to P_{\mathbf{i}}$ by the formula $x(\sum t_\gamma u_\gamma) = \sum x(u_\gamma) t_\gamma$. (The sum is finite by the definition of $\mathcal{A}$.)

For every family $\mathbf{E}$ as above define a map $\psi_{\mathbf{i}} = \psi_{\mathbf{i},\mathbf{E}} : \mathcal{A} \to P_{\mathbf{i}}$ by the formula $\psi_{\mathbf{i}}(x) := x(\mathbf{e}_{\mathbf{i}})$. Expanding $\mathbf{e}_{\mathbf{i}}$ into a power series we obtain

$$\psi_{\mathbf{i}}(x) = \sum_{a_1, \ldots, a_m \in \mathbf{Z}_+} x\big(E_1^{[a_1]} E_2^{[a_2]} \cdots E_m^{[a_m]}\big) t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m} \tag{1.7}$$

where $E_k^{[n]} = \dfrac{E_k^n}{[n]_{q_k}!}$. Note that the sum in (1.7) is always finite because x vanishes on all but finitely many monomials $E_1^{a_1} \cdots E_{i_m}^{a_m}$. Define a $\mathbf{Z}_+^r$-grading on $P_{\mathbf{i}}$ by $\deg(t_k) = \alpha_{i_k}$ and denote by $P_{\mathbf{i}}(\gamma)$ the graded component of degree γ in $P_{\mathbf{i}}$.

Corollary 1.3. *For any sequence $\mathbf{i} = (i_1, \ldots, i_m)$ and a family $\mathbf{E}$ of elements $E_k \in \mathcal{U}(\alpha_{i_k})$ $(k = 1, \ldots, m)$, the map $\psi_{\mathbf{i},\mathbf{E}} : \mathcal{A} \to P_{\mathbf{i}}$ defined by (1.7) is a homomorphism of $\mathbf{Z}_+^r$-graded algebras.*

The proof of Corollary 1.3 repeats that of Corollary 0.2. $\triangleleft$

Remark 1. One can prove (see e.g. [10]) that $\mathcal{A}$ is a $\mathbf{q}^t$-braided bialgebra (where $\mathbf{q}^t$ is the transpose of $\mathbf{q}$). Moreover, starting with an arbitrary $\mathbf{q}^t$-braided algebra $\mathcal{A}$, one recovers $\mathcal{U}$ as the restricted dual of $\mathcal{A}$. So the result of Corollary 1.2 holds for any $\mathbf{q}^t$-braided bialgebra $\mathcal{A}$.

Remark 2. Let $\mathcal{A}_1 = \bigoplus_{i=1}^{r} \mathcal{A}(\alpha_i)$. Corollary 1.2 implies that any morphism $\mathcal{A}_1 \to \bigoplus_{i=1}^{r} P(\alpha_i)$ of $\mathbf{Z}_+^r$-graded vector spaces extends to an algebra homomorphism. If $\mathcal{A}$ is generated by $\mathcal{A}_1$, then this extension is unique. Thus, in the latter case all the homomorphisms $\mathcal{A} \to P_{\mathbf{i}}$ of $\mathbf{Z}_+^r$-graded algebras are parametrized by the space $\bigoplus_{i=1}^{r} \big(\mathcal{U}(\alpha_i) \bigotimes P_{\mathbf{i}}(\alpha_i)\big)$.

We define the *universal* element $\mathcal{R} \in \mathcal{A} \bigotimes \hat{\mathcal{U}}$ as follows. The tensor product $\mathcal{A} \bigotimes \hat{\mathcal{U}}$ is canonically identified with the space of all linear maps $\mathcal{U} \to \hat{\mathcal{U}}$. Then $\mathcal{R}$ is the element in $\mathcal{A} \bigotimes \hat{\mathcal{U}}$ corresponding to the inclusion $\mathcal{U} \hookrightarrow \hat{\mathcal{U}}$.

Proposition 1.4. *The element $\mathcal{R}$ satisfies $(\psi_{\mathbf{i},\mathbf{E}} \otimes \mathrm{id})(\mathcal{R}) = \mathbf{e}_{\mathbf{i},\mathbf{E}}$ for any $\mathbf{i}$ and $\mathbf{E}$ as above.*

The proof of Proposition 1.4 coincides with that of Proposition 0.3. $\triangleleft$

Clearly, the correspondence

$$c_1t_1 + \cdots + c_mt_m \mapsto \exp_{q_1}(c_1t_1E_1)\cdots\exp_{q_m}(c_mt_mE_m)$$

is a map from the m-dimensional "quantum" affine space $(P_{\mathbf{i}})_1 = \oplus_{l=1}^{m} \mathbf{k} \cdot t_l$ to the set of group-like elements in $\hat{\mathcal{U}_{\mathbf{i}}}$. This map can be regarded as a deformation of the morphism (0.1).

Now let us turn to the fields of fractions. For an algebra $\mathcal{B}$ without zero divisors, $\mathcal{F}(\mathcal{B})$ is a vector space of right fractions (see Appendix). Its elements can be written as ab^{-1}, where $a, b \in \mathcal{B}$ and $b \neq 0$. As shown in the Appendix, for any sequence $\mathbf{i}$ and any subalgebra $\mathcal{B} \subset P_{\mathbf{i}}$, the space $\mathcal{F}(\mathcal{B})$ is a skew-field. Note that $\psi_{\mathbf{i}}$ induces an embedding of skew fields $\overline{\psi}_{\mathbf{i}} : \mathcal{F}(\mathcal{A}/\mathrm{Ker}\ \psi_{\mathbf{i}}) \hookrightarrow \mathcal{F}(P_{\mathbf{i}})$.

Now consider two elements $\mathbf{e}_{\mathbf{i},\mathbf{E}}$ and $\mathbf{e}_{\mathbf{i}',\mathbf{E}'}$ corresponding via (1.6) to two sequences of indices $\mathbf{i} = (i_1, \ldots, i_m)$ and $\mathbf{i}' = (i'_1, \ldots, i'_n)$ and two families of elements $\mathbf{E} = (E_1, \ldots, E_m)$ and $\mathbf{E}' = (E'_1, \ldots, E'_n)$. Let $t_1, \ldots, t_m$ (resp. $t'_1, \ldots, t'_n$) be the standard generators of $P_{\mathbf{i}}$ (resp. $P_{\mathbf{i}'}$).

Proposition 1.5. *Assume that* Ker $\psi_{\mathbf{i},\mathbf{E}}$ = Ker $\psi_{\mathbf{i}',\mathbf{E}'}$, *and $\overline{\psi}_{\mathbf{i}',\mathbf{E}'}$ is an isomorphism of skew-fields $\mathcal{F}(\mathcal{A}/\mathrm{Ker}\ \psi_{\mathbf{i}',\mathbf{E}'})$ and $\mathcal{F}(P_{\mathbf{i}'})$. Then the map $\mathbf{R} := \overline{\psi}_{\mathbf{i},\mathbf{E}} \circ (\overline{\psi}_{\mathbf{i}',\mathbf{E}'})^{-1}$ is an embedding $\mathcal{F}(P_{\mathbf{i}'}) \hookrightarrow \mathcal{F}(P_{\mathbf{i}})$, and the following identity holds in $\mathcal{F}(P_{\mathbf{i}}) \bigotimes \hat{\mathcal{U}}$:*

$$\exp_{q_1}(t_1E_1)\cdots\exp_{q_m}(t_mE_m) = \exp_{q'_1}(p_1E'_1)\cdots\exp_{q'_n}(p_nE'_n)\ , \tag{1.8}$$

where $q_k = q_{i_k,i_k}$ for $k = 1, \ldots, m$, and $q'_l = q_{i'_l,i'_l}$, $p_l = \mathbf{R}(t'_l)$ for $l = 1, \ldots, n$.

Proof. We omit subscripts $\mathbf{E}$ and $\mathbf{E}'$ in the formulas below. Let $\mathbf{p}_{\mathbf{i}} : \mathcal{A} \to \mathcal{A}/\mathrm{Ker}\ \psi_{\mathbf{i}}$ be the canonical projection. Denote $\mathcal{R}_{\mathbf{i}} = (\mathbf{p}_{\mathbf{i}} \otimes \mathrm{id})(\mathcal{R})$. Then Proposition 1.4 implies that

$$(\overline{\psi}_{\mathbf{i}} \otimes \mathrm{id})(\mathcal{R}_{\mathbf{i}}) = \mathbf{e}_{\mathbf{i}} \tag{1.9}$$

for every $\mathbf{i} \in R(w)$. We are done since $\mathbf{R} \otimes \mathrm{id} = (\overline{\psi}_{\mathbf{i}} \otimes \mathrm{id}) \circ (\overline{\psi}_{\mathbf{i}'} \otimes \mathrm{id})^{-1}$. Proposition 1.5 is proved. $\triangleleft$

2. Feigin's conjecture and other results for quantum groups

Throughout this section we will work over the field $\mathbf{k} = k(q)$ where k is a field of characteristic 0 (say, $k = \mathbf{C}$ as in the introduction), and q is a variable (or a purely transcendental element over k). Let $A = (a_{ij})$ be a symmetrizable Cartan matrix of size $r \times r$, and $C = (C_{ij})$ be the corresponding symmetric matrix with integer entries. In this section we consider a matrix $\mathbf{q}$ of the form $\mathbf{q} = (q_{ij}) = (q^{C_{ij}})$. We denote $q_i := q_{ii} = q^{C_{ii}}$ for all i.

Similarly to [8], Chapter 1, we define the *quantized enveloping algebra* $\mathcal{U}$ and the *quantum group* $\mathcal{A}$ associated with A as follows. First, let $\overline{\mathcal{U}}$ be the free algebra over $k(q)$ generated by $E_1, \ldots, E_r$. We make $\overline{\mathcal{U}}$ into a $\mathbf{q}$-braided bialgebra (see Section 1). Second, the restricted dual algebra $\overline{\mathcal{A}}$ of $\overline{\mathcal{U}}$ is defined as in Section 1. Define a homomorphism $f : \overline{\mathcal{U}} \to \overline{\mathcal{A}}$ by $f(E_i) = x_i$ where x_i is the only element in $\overline{\mathcal{A}}(\alpha_i)$ such that $x_i(E_i) = 1$. Finally, define $\mathcal{U} := \overline{\mathcal{U}}/\mathrm{Ker}\ f$ and $\mathcal{A} := \mathrm{Im}\ f$, and keep the above notation for the generators. In particular, $\mathcal{U} \cong \mathcal{A}$ via $E_i \mapsto x_i$. It is well-known that the right kernel of the evaluation pairing $\mathcal{A} \otimes \overline{\mathcal{U}} \to k(q)$ coincides with Ker f. Hence the induced pairing

$$\mathcal{A} \otimes \mathcal{U} \to k(q) \tag{2.1}$$

is non-degenerate, so we identify $\mathcal{A}$ with the restricted dual algebra to the $\mathbf{q}$-braided bialgebra $\mathcal{U}$ (and denote the evaluation pairing (2.1) by $(x, E) \mapsto x(E)$). Note that the generators $E_1, \ldots, E_r$ of $\mathcal{U}$ (as well as the generators $x_1, \ldots, x_r$ of $\mathcal{A}$) are subject to the quantum Serre relations ([8], Section 1.4.3, or Section 3 below).

The algebra $\mathcal{U}$ is $\mathbf{Z}_+^r$-graded via deg $E_i = \alpha_i$. The pairing $\mathcal{A} \times \mathcal{U} \to k(q)$ extends to the $P_{\mathbf{i}}$-linear pairing $\mathcal{A} \times \hat{\mathcal{U}}_{\mathbf{i}} \to k(q)$ (we denote it by $(x, u) \mapsto x(u)$), where $\hat{\mathcal{U}}_{\mathbf{i}} := P_{\mathbf{i}} \bigotimes \hat{\mathcal{U}}$ and $P_{\mathbf{i}}$ is a $k(q)$-algebra generated by $t_1, \ldots, t_m$ subject to the relations $t_l t_k = q^{C_{i_k, i_l}} t_k t_l$ for $1 \leq k < l \leq m$.

For the convenience of the reader, we summarize the results from Section 1 for the quantum groups $\mathcal{A}$ and $\mathcal{U}$ in the following theorem.

Theorem 2.1. *Let $\mathbf{i} = (i_1, \ldots, i_m)$ be any sequence. Then*

(a) the element

$$\mathbf{e}_{\mathbf{i}} = \exp_{q_{i_1}}(t_1 E_{i_1}) \cdots \exp_{q_{i_m}}(t_m E_{i_m})$$

is a group-like element in $\hat{\mathcal{U}}_{\mathbf{i}}$;

(b) the element $\mathbf{e}_{\mathbf{i}}$ gives rise to an algebra homomorphism $\psi_{\mathbf{i}} : \mathcal{A} \to P_{\mathbf{i}}$ defined by $\psi_{\mathbf{i}}(x) := x(\mathbf{e}_{\mathbf{i}})$;

(c) there is a unique element $\mathcal{R} \in \mathcal{A} \bigotimes \hat{\mathcal{U}}$ satisfying $(\psi_{\mathbf{i}} \otimes \mathrm{id})(\mathcal{R}) = \mathbf{e}_{\mathbf{i}}$ for all $\mathbf{i}$;

(d) the homomorphism $\psi_{\mathbf{i}}$ satisfies

$$\psi_{\mathbf{i}}(x) = \sum_{a_1,\ldots,a_m \geq 0} x(E_{i_1}^{[a_1]} E_{i_2}^{[a_2]} \cdots E_{i_m}^{[a_m]}) t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m} \ , \tag{2.2}$$

where $E_i^{[n]} = \frac{1}{[n]_{q_i}!} E_i^n$, and $[n]_q!$ is defined in (0.4). In particular, for $i = 1, \ldots, r$ we have $\psi_{\mathbf{i}}(x_i) = \sum_{k: i_k = i} t_k$, and this determines $\psi_{\mathbf{i}}$ uniquely;

(e) Ker $\psi_{\mathbf{i}} = \{x \in \mathcal{A} : x(E_{i_1}^{a_1} E_{i_2}^{a_2} \cdots E_{i_m}^{a_m}) = 0$ for all $a_1, \ldots, a_m \in \mathbf{Z}_+\}$.

Remark. After the identification $\mathcal{A} \cong \mathcal{U}$ as above, $\psi_{\mathbf{i}}$ coincides with Feigin's homomorphism $\Phi(\mathbf{i}) : \mathcal{U} \to P_{\mathbf{i}}$. B. Feigin introduced this homomorphism in his talk at RIMS in 1992 (see e.g. [5] and [6]).

Let W be the Weyl group associated with the Cartan matrix A. By definition, W is generated by simple reflections $s_1, \ldots, s_r : \mathbf{Z}^r \to \mathbf{Z}^r$ where $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. We call a sequence $\mathbf{i} = (i_1, \ldots, i_m)$ of indices a *reduced expression* of $w \in W$ if $w = s_{i_1} s_{i_2} \cdots s_{i_m}$, and the above expression of w is the shortest (we call $\mathbf{i}$ simply a reduced expression if w is not specified). We set $l(w) := m$ and call $l(w)$ the *length* of w. Denote by $R(w)$ the set of all reduced expressions of w. It is well-known that W is a Coxeter group, so the defining relations between $s_1, \ldots, s_r$ are of the form $(s_i s_j)^l = 1$ where $l \in \{2, 3, 4, 6\}$. (More precisely, $l = a_{ij}a_{ji} + 2$ if $a_{ij}a_{ji} < 3$ and $l = 6$ if $a_{ij}a_{ji} = 3$.) It follows that every two reduced expressions of an element $w \in W$ are connected by a chain of moves

$$(\mathbf{i}_1, (i, j, i, \ldots), \mathbf{i}_2) \mapsto (\mathbf{i}_1, (j, i, j \ldots), \mathbf{i}_2)$$

where each fragment in parentheses has the length l. If the Weyl group W is finite then there is a unique element of the maximal length in W which we denote by w_0.

Let us study the kernel of $\psi_{\mathbf{i}}$. According to Theorem 2.1(e), Ker $\psi_{\mathbf{i}}$ is the orthogonal complement of the subspace $\mathcal{U}(\mathbf{i}) \subset \mathcal{U}$ spanned by all monomials $E_{i_1}^{a_1} E_{i_2}^{a_2} \cdots E_{i_m}^{a_m}$.

Lemma 2.2.

(a) For every sequence $\mathbf{i}$ there is a reduced expression $\mathbf{i}'$ such that $\mathcal{U}(\mathbf{i}) = \mathcal{U}(\mathbf{i}')$. Moreover, $\mathbf{i}'$ can always be chosen as a subsequence of $\mathbf{i}$.

(b) For any $w \in W$ and $\mathbf{i}, \mathbf{i}' \in R(w)$, we have $\mathcal{U}(\mathbf{i}) = \mathcal{U}(\mathbf{i}')$.

(c) $\mathcal{U}(\mathbf{i})$ contains the subalgebra in $\mathcal{U}$ generated by all E_i such that $l(ws_i) = l(w) - 1$. Therefore, $\mathcal{U}(\mathbf{i}) = \mathcal{U}$ for every $\mathbf{i} \in R(w_0)$ when W is finite.

Proof. The collection of the subspaces $\{\mathcal{U}(\mathbf{i})\}$ is a multiplicative semigroup with respect to the product of vector subspaces in $\mathcal{U}$. By definition,

$$\mathcal{U}(i_1, i_2, \ldots, i_m) = \mathcal{U}(i_1) \cdot \mathcal{U}(i_2) \cdots \mathcal{U}(i_m)$$

where $\mathcal{U}(i)$ is a subalgebra in $\mathcal{U}$ generated by E_i, $i = 1, \ldots, r$.

We have $\mathcal{U}(i)\mathcal{U}(i) = \mathcal{U}(i)$, and for every pair (i, j) with $a_{ij}a_{ji} < 4$ the following relation holds:

$$\mathcal{U}(i) \cdot \mathcal{U}(j) \cdot \mathcal{U}(i) \cdots = \mathcal{U}(j) \cdot \mathcal{U}(i) \cdot \mathcal{U}(j) \cdots \tag{2.3}$$

where each product contains l factors. The identity (2.3) can be proved by the standard arguments for the algebras $\mathcal{U}$ whose Cartan matrices are of types $A_1 \times A_1, A_2, B_2$ or G_2.

It follows that every $\mathcal{U}(\mathbf{i})$ equals to $\mathcal{U}(\mathbf{i}')$ for some reduced subsequence $\mathbf{i}'$ of $\mathbf{i}$, which proves (a). Part (b) also follows because the braid relations (2.3) can be used to move from any reduced expression of $w \in W$ to any other one.

(c) Let $J = J_w$ be the the set of all i satisfying $l(ws_i) = l(w) - 1$. For each $i \in J$, there exists $\mathbf{i} \in R(w)$ such that $\mathbf{i}$ ends with i. Using (b) we see that $\mathcal{U}(\mathbf{i})E_i \subset \mathcal{U}(\mathbf{i})$ for any $\mathbf{i} \in R(w)$, $i \in J$. This completes the proof of Lemma 2.2. $\triangleleft$

We define $I_w :=$ Ker $\psi_{\mathbf{i}}$ for any $\mathbf{i} \in R(w)$. Since $\mathcal{A}/I_w$ is isomorphic to $\psi_{\mathbf{i}}(\mathcal{A})$, it follows that $\mathcal{F}(\mathcal{A}/I_w)$ is a skew field (see Appendix).

Theorem 2.3. *For every $w \in W$ and $\mathbf{i} \in R(w)$ the map $\psi_{\mathbf{i}}$ induces an isomorphism of skew fields*

$$\overline{\psi}_{\mathbf{i}} : \mathcal{F}(\mathcal{A}/I_w) \cong \mathcal{F}(P_{\mathbf{i}}) \ . \tag{2.4}$$

Taking $w = w_0$ we obtain the following

Corollary 2.4. *(Feigin's conjecture). For any $\mathbf{i} \in R(w_0)$, the homomorphism $\psi_{\mathbf{i}} : \mathcal{A} \to P_{\mathbf{i}}$ is an embedding, and it induces an isomorphism of skew-fields*

$$\overline{\psi}_{\mathbf{i}} : \mathcal{F}(\mathcal{A}) \cong \mathcal{F}(P_{\mathbf{i}}) \ .$$

Proof of Theorem 2.3. It is enough to prove that for any $\mathbf{i} \in R(w)$ the image of $\psi_{\mathbf{i}}$ generates $\mathcal{F}(P_{\mathbf{i}})$, that is, $t_1, \ldots, t_m$ belong to $\mathcal{F}(\text{Im } \psi_{\mathbf{i}})$. We will deduce this statement from Proposition 0.6. Then we neeed the following.

Proposition 2.5. *For each element $x \in \mathcal{A}$ satisfying*

$$\psi_{\mathbf{i}}(x) = t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m} \tag{2.5}$$

with $a_1 > 0$, there is an element $y \in \mathcal{A}$ such that $\psi_{\mathbf{i}}(y) = ct_1^{a_1-1}t_2^{a_2} \cdots t_m^{a_m}$ where $c \in k(q), c \neq 0$.

Proof of Proposition 2.5. For $i = 1, \ldots, r$, let $E_i^* : \mathcal{A} \to \mathcal{A}$ be the adjoint operator of the left multiplication operator $E \mapsto E_iE$ in $\mathcal{U}$. Thus, the element $E_i^*(x)$ is determined by the equations $(E_i^*(x))(E) = x(E_iE)$ for every $E \in \mathcal{U}$.

We will show that y can be chosen as $y = E_i^*(x)$.

Indeed, (2.5) means that the right hand side of the expansion (2.2) for $\psi_{\mathbf{i}}(x)$ reduces to one summand or, equivalently,

$$x(E_{i_1}^{[b_1]} \cdots E_{i_m}^{[b_m]}) = 0 \tag{2.6}$$

unless $(b_1, \ldots, b_m) = (a_1, \ldots, a_m)$.

By (2.2), we have

$$\psi_{\mathbf{i}}(y) = \psi_{\mathbf{i}}(E_{i_1}^*(x)) = \sum_{b_1,\ldots,b_m \in \mathbf{Z}_+} (E_{i_1}^*(x))\big(E_{i_1}^{[b_1]} E_{i_2}^{[b_2]} \cdots E_{i_m}^{[b_m]}\big) t_1^{b_1} t_2^{b_2} \cdots t_m^{b_m}$$

$$= \sum_{b_1,\ldots,b_m \in \mathbf{Z}_+} x\big(E_{i_1} E_{i_1}^{[b_1]} E_{i_2}^{[b_2]} \cdots E_{i_m}^{[b_m]}\big) t_1^{b_1} t_2^{b_2} \cdots t_m^{b_m} \ .$$

In view of (2.6),

$$\psi_{\mathbf{i}}(y) = x(E_{i_1} E_{i_1}^{[a_1-1]} E_{i_2}^{[a_2]} \cdots E_{i_m}^{[a_m]}) t_1^{a_1-1} t_2^{a_2} \cdots t_m^{a_m} = c t_1^{a_1-1} t_2^{a_2} \cdots t_m^{a_m}$$

with $c \neq 0$ as desired. $\triangleleft$

Taking x and y as in Proposition 2.5, we see that

$$t_1 = c\psi_{\mathbf{i}}(x)(\psi_{\mathbf{i}}(y))^{-1} \in \ \mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}}) \ . \tag{2.7}$$

To complete the proof of Theorem 2.3, we proceed by induction on m. If $m = 1$ then $t_1 \in \mathrm{Im}\ \psi_{\mathbf{i}} = P_{\mathbf{i}}$. So let $m \geq 2$, denote $\mathbf{i}' = (i_2, \ldots, i_m)$ and assume that Theorem 2.3 holds for $\mathbf{i}'$, that is,

$$t_2, t_3, \ldots, t_m \in \mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}'}) \ . \tag{2.8}$$

Note that $P_{\mathbf{i}'}$ is naturally embedded into $P_{\mathbf{i}}$ as a subalgebra generated by $t_2, \ldots, t_m$. In view of the formula for $\psi_{\mathbf{i}}(x_i)$ in Theorem 2.1(d),

$$\psi_{\mathbf{i}'}(x_i) = \psi_{\mathbf{i}}(x_i) \ (i \neq i_1), \ \ \psi_{\mathbf{i}'}(x_{i_1}) = \psi_{\mathbf{i}}(x_{i_1}) - t_1 \ . \tag{2.9}$$

Using (2.7). we see that

$$\psi_{\mathbf{i}'}(x_i) \in \mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}}) \ , (i = 1, \ldots, r)$$

hence

$$\mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}'}) \subset \mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}}) \ .$$

Combining this with the inductive assumption (2.8), we conclude that $t_2, \ldots, t_m \in \mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}})$. Since t_1 also belongs to $\mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}})$, Theorem 2.3 is proved. $\triangleleft$

Proof of Proposition 0.8. let $\mathcal{B}$ be the algebra of the upper triangular $n \times n$-matrices over $\mathbf{C}(q)$ (with the unity I, the identity matrix). Let $\rho : \mathcal{U} \to \mathcal{B}$ be a representation of $\mathcal{U}$ given by $\rho(E_i) = E_{i,i+1}$, where E_{ij} is the matrix unit. The representation ρ extends naturally to $\mathrm{id} \otimes \rho : \mathcal{A} \bigotimes \mathcal{U} \to \mathcal{A} \bigotimes \mathcal{B}$. We identify the latter algebra with $\mathcal{B}(\mathcal{A})$, the algebra of upper triangular matrices over $\mathcal{A}$.

Lemma 2.6. *We have $X = (\mathrm{id} \otimes \rho)(\mathcal{R})$, where $\mathcal{R}$ is the universal element in $\mathcal{A} \bigotimes \hat{\mathcal{U}}$.*

Proof. Note that $\mathcal{B}$ is a $\mathbf{Z}_+^{n-1}$-graded algebra via $\deg(E_{ii}) = 0$, $\deg(E_{ij}) = \alpha_{ij} = \alpha_i + \cdots \alpha_{j-1}$ $(1 \le i < j \le n)$, and ρ preserves the $\mathbf{Z}_+^{n-1}$-grading $(n = r+1)$. Therefore, the formula (0.9) for $\mathcal{R}$ implies

$$(\mathrm{id} \otimes \rho)(\mathcal{R}) = I + \sum_{i<j} \sum_{b \in B_{ij}} b^* \rho(b) \ .$$

where B_{ij} is a basis in $\mathcal{U}(\alpha_{ij})$ and $\{b^*\}$ is the dual basis in $\mathcal{A}(\alpha_{ij})$. We choose B_{ij} to consist of the products (in any order) of the generators $E_i, E_{i+1}, \ldots, E_{j-1}$. It is easy to see that $\rho(b) = E_{ij}$ for the element $b = b_{ij} = E_i E_{i+1} \cdots E_{j-1}$ in B_{ij}, and $\rho(b) = 0$ if $b \in B_{ij}, b \ne b_{ij}$. Denote $x_{ij} = (b_{ij})^*$. To identify these x_{ij} with those in Section 0 we have to verify the relations (0.11). As an algebra, $\mathcal{A}$ is generated by the $x_i := x_{i,i+1}$ $(i = 1, \ldots, r)$ subject to the quantum Serre relations. The relations (0.11) can be verified similarly to those between the t_{ij} in [2], Section 3 (they also follow from the relations in [4]). Thus, $(\mathrm{id} \otimes \rho)(\mathcal{R}) = I + \sum\limits_{i<j} x_{ij} E_{ij} = X$. Lemma 2.6 is proved. $\triangleleft$

To complete the proof of Proposition 0.8, note that, for every $\mathbf{i}$ we have $\psi_{\mathbf{i}}(X) = (\psi_{\mathbf{i}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \rho)(\mathcal{R}) = (\mathrm{id} \otimes \rho) \circ (\psi_{\mathbf{i}} \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes \rho)(\mathbf{e_i})$, by (0.8). The formula (0.12) follows since $(\mathrm{id} \otimes \rho)\big(\exp_{q_{i_k}}(t_k E_{i_k})\big) = I + t_k E_{i_k, i_k+1}$ for all k. Proposition 0.8 is proved. $\triangleleft$

We have the following obvious corollary of Theorem 2.3.

Corollary 2.7. *For any $w \in W$ and $\mathbf{i}, \mathbf{i'} \in R(w)$ there is an isomorphism of skew-fields*

$$R_{\mathbf{i}}^{\mathbf{i'}} : \mathcal{F}(P_{\mathbf{i}}) \widetilde{\to} \mathcal{F}(P_{\mathbf{i'}}) \tag{2.10}$$

defined by $R_{\mathbf{i}}^{\mathbf{i'}} := \overline{\psi}_{\mathbf{i'}} \circ (\overline{\psi}_{\mathbf{i}})^{-1}$.

The "transition maps" $R_{\mathbf{i}}^{\mathbf{i'}}$ lead to identities between quantum exponentials given by Corollary 0.10. To compute each p_k in (0.14), it is enough (in principle) to do this for the following pairs:

$$\mathbf{i} = (i, j, i \ldots), \ \mathbf{i'} = (j, i, j, \ldots) \ ,$$

of the length l each, where l is the order of $s_i s_j$ in W. Recall that $l = 2, 3, 4$, or 6 for any Weyl group. In the following proposition, we compute $\mathbf{R}_{\mathbf{i'}}^{\mathbf{i}}$ for these $\mathbf{i}, \mathbf{i'}$ with $l = 2, 3$, or 4 (when $l = 6$ the explicit expressions for p_k are more complicated, so we do not present them here).

Proposition 2.8. *Let $t_1, \ldots, t_l$ (resp. $t'_1, \ldots, t'_l$) be standard generators of $P_{(iji\ldots)}$ (resp. $P_{(jij\ldots)}$). We denote $p_k = \mathbf{R}_{(jij\ldots)}^{(iji\ldots)}(t'_k)$ $(k = 1, \ldots, l)$.*

(a) *If* $l = 2$ *then* $(p_1, p_2) = (t_2, t_1)$.
(b) *If* $l = 3$ *then* $(p_1, p_2, p_3) = \big(t_2 t_3 (t_1 + t_3)^{-1},\ t_1 + t_3,\ (t_1 + t_3)^{-1} t_1 t_2\big)$.
(c) *If* $l = 4$ *and* $a_{ij} = -2, a_{ji} = -1$ *then* p_1, p_2, p_3, p_4 *are determined by the following equations:*

$$p_2 p_3 = t_1 t_2 + t_1 t_4 + t_3 t_4, \quad p_2 p_3 p_4 = t_1 t_2 t_3 \ ,$$

$$p_2^2 p_3 = t_1^2 t_2 + (t_1 + t_3)^2 t_4, \quad p_1 p_2^2 p_3 = t_2 t_3^2 t_4 \ .$$

Proof. (a) By definition, $\psi_{(ij)} : (x_i, x_j) \mapsto (t_1, t_2)$ and $\psi_{(ji)} : (x_i, x_j) \mapsto (t_2', t_1')$. Thus, $p_1 = t_2$, and $p_2 = t_1$ as claimed.

Part (b) is proved in Section 0 (see Example 1).

(c) Let $\mathcal{U}_{ij}$ be the subalgebra of $\mathcal{U}$ generated by E_i and E_j. Define $\mathcal{B}_{ij}$ as the quotient algebra of $\mathcal{U}_{ij}$ modulo the relations $E_i^3 = E_j^2 = 0$ (we keep the same notation for generators). It is easy to see ([8], or Section 3 below) that the following are all the defining relations in $\mathcal{B}_{ij}$:

$$E_i^3 = E_j^2 = E_j E_i E_j = 0, \ E_i^2 E_j E_i = E_i E_j E_i^2 \ .$$

Using these relations, it is easy to prove that the homogeneous components $\mathcal{B}_{ij}(\alpha_i + \alpha_j)$, $\mathcal{B}_{ij}(2\alpha_i + \alpha_j)$, and $\mathcal{B}_{ij}(2\alpha_i + 2\alpha_j)$ of $\mathcal{B}_{ij}$ have the following bases: $\{E_i E_j,\ E_j E_i\}$ for $\mathcal{B}_{ij}(\alpha_i + \alpha_j)$, $\{E_i^2 E_j,\ E_i E_j E_i,\ E_j E_i^2\}$ for $\mathcal{B}_{ij}(2\alpha_i + \alpha_j)$, and $\{E_j E_i^2 E_j\}$ for $\mathcal{B}_{ij}(2\alpha_i + 2\alpha_j)$.

Applying the projection $\rho : \mathcal{U}_{ij} \to \mathcal{B}_{ij}$ to both sides of (0.14), we obtain the following equation in $\mathcal{F}(P_{ijij}) \bigotimes \mathcal{B}_{ij}$:

$$(1 + p_1 E_j)(1 + p_2 E_i + \frac{p_2^2 E_i^2}{1 + q^2})(1 + p_3 E_j)(1 + p_4 E_i + \frac{p_4^2 E_i^2}{1 + q^2})$$

$$= (1 + t_1 E_i + \frac{t_1^2 E_i^2}{1 + q^2})(1 + t_2 E_j)(1 + t_3 E_i + \frac{t_3^2 E_i^2}{1 + q^2})(1 + t_4 E_j) \ . \tag{2.11}$$

The desired expressions for $p_2 p_3,\ p_2 p_3 p_4,\ p_2^2 p_3$, and $p_1 p_2^2 p_3$ can be obtained from (2.11) by comparing the coefficients of $E_i E_j,\ E_i E_j E_i,\ E_i^2 E_j$, and $E_j E_i^2 E_j$ respectively on both sides of (2.11). Proposition 2.9 is proved. $\triangleleft$

Remark. Taking in the identities of Proposition 2.8 the homogeneous components of degrees $\alpha_i + (1 - a_{ij})\alpha_j$ and $\alpha_j + (1 - a_{ji})\alpha_i$ yields quantum Serre relations between E_i and E_j.

We conclude this section by a proof of Proposition 0.11. For a skew-symmetric $m \times m$-matrix $S = (S_{kl})$ with integer entries let P_S be a $k(q)$-algebra generated by $t_1, \ldots, t_m$ subject to the relations

$$t_l t_k = q^{S_{kl}} t_k t_l \ . \tag{2.12}$$

Note that $P_{\mathbf{i}} = P_{S(\mathbf{i})}$ where the matrix $S(\mathbf{i})$ is defined in (0.16). Recall that two $m \times m$-matrices S and S' are called equivalent if there is a matrix $T = T_{kl} \in SL_m(\mathbf{Z})$ such that

$S' = TST^t$. It is easy to see that $\mathcal{F}(P_S) \cong \mathcal{F}(P_{S'})$ if S and S' are equivalent: one can choose such an isomorphism $\mathcal{F}(P_{S'}) \widetilde{\to} \mathcal{F}(P_S)$ by sending each generator t'_k of $P_{S'}$ to the monomial $t_1^{T_{k,1}} t_2^{T_{k,2}} \cdots t_m^{T_{k,m}}$ in the generators of P_S. The converse statement was proved by A. Panov.

Proposition 2.9 ([11], Theorem 2.19). *Let S, S' be skew-symmetric $m \times m$ matrices with the integer entries. Then $\mathcal{F}(P_S) \cong \mathcal{F}(P_{S'})$ if and only if S and S' are equivalent.*

Thus, Proposition 2.9 means that the existence of *any* isomorphism $\mathbf{R} : \mathcal{F}(P_{S'}) \to \mathcal{F}(P_S)$ implies that of a *monomial* isomorphism $\mathbf{M} : \mathcal{F}(P_{S'}) \to \mathcal{F}(P_S)$, that is, $\mathbf{M}$ takes each generator t'_k of $P_{S'}$ to a monomial in generators $t_1, \ldots, t_m$ of P_S. Taking $S = S(\mathbf{i}), S' = S(\mathbf{i}')$, and $\mathbf{R} = \mathbf{R}_{\mathbf{i}'}^{\mathbf{i}} : \mathcal{F}(P_{\mathbf{i}'}) \to \mathcal{F}(P_{\mathbf{i}})$ with $\mathbf{i}, \mathbf{i}' \in R(w)$ for some $w \in W$, we obtain, in particular, the statement of Proposition 0.11. Note that $\mathrm{R}_{\mathbf{i}'}^{\mathbf{i}}$ is not monomial in general.

One can prove that there exists a *local* monomial isomorphism $\mathbf{M} = \mathbf{M}_{\mathbf{i},\mathbf{i}'} : \mathcal{F}(P_{\mathbf{i}'}) \to \mathcal{F}(P_{\mathbf{i}})$. Namely, for $\mathbf{i}, \mathbf{i}'$ of the form

$$\mathbf{i} = (i_1, \ldots, i_{a-l}; i, j, i, \ldots; i_{a+1}, \ldots, i_m)$$

$$\mathbf{i}' = (i_1, \ldots, i_{a-l}; j, i, j, \ldots; i_{a+1}, \ldots, i_m) \ ,$$

$\mathbf{M}(t'_k) = t_k$ if $k \leq a - l$ or $k > a$, and each $\mathbf{M}(t'_k)$ for $k = a - l + 1, \ldots, a$ depends only on $t_{a-l+1}, \ldots, t_a$. We will present such $\mathbf{M}$ elswhere.

By Proposition 2.9, the equivalence class of $S(\mathbf{i})$ for $\mathbf{i} \in R(w)$ depends only on w. If we choose some representative $\mathbf{S}(w)$ of this class then, by Theorem 2.3, there is an isomorphism

$$\mathcal{F}\big(\mathcal{A}/I_w\big) \widetilde{\to} \mathcal{F}(P_{\mathbf{S}(w)}) \ . \tag{2.16}$$

The well-known normal form for skew-symmetric matrices shows that $\mathbf{S}(w) = (\mathbf{S}_{kl})$ can be chosen uniquely subject to the following requirements:

(i) $\mathbf{S}_{kl} = 0$ unless $k + l \neq m + 1$;

(ii) there is a sequence $c_1, c_2, \ldots$ of nonnegative integers such that $\mathbf{S}_{k,m+1-k} = c_1 c_2 \cdots c_k$ for $1 \leq k \leq \frac{m}{2}$.

It would be interesting to compute the invariants $c_1, \ldots, c_m$ in terms of w, and to find a direct way to describe the isomorphism (2.16).

3. Extremal vectors in $\mathcal{A}$ and proof of Proposition 0.6

We retain terminology and notation of Section 2. Recall that $\alpha_1, \ldots, \alpha_r$ is the standard basis in $\mathbf{Z}_+^r$. We define a bilinear form in $\mathbf{Z}^r$ by the formula $(\alpha_i, \alpha_j) = C_{ij}$ for all i, j.

Let us fix $\lambda = (l_1, \ldots, l_r) \in \mathbf{Z}_+^r$. For $i = 1, \ldots, r$ define linear operators $F_i = F_{i,\lambda} : \mathcal{A} \to \mathcal{A}$ by the formula:

$$F_i \cdot x = \frac{v_i^{l_i} q^{-(\gamma,\alpha_i)} x x_i - v_i^{-l_i} x_i x}{v_i - v_i^{-1}} \tag{3.1}$$

for $x \in \mathcal{A}(\gamma)$ (where $v_i = q^{\frac{C_{ii}}{2}}$).

We identify λ with a linear form on the *coroot* lattice $\mathbf{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbf{Z}\alpha_r^\vee$ defined by $\lambda(\alpha_i^\vee) := l_i$ (recall that $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$). For each reduced $\mathbf{i} = (i_1, \ldots, i_m)$ we define a sequence of integers $a_1, \ldots, a_m$ by the formula

$$a_1 = \lambda(\alpha_{i_1}^\vee), \quad a_2 = \lambda(s_{i_1}(\alpha_{i_2}^\vee)), \ldots, a_m = \lambda\big(s_{i_1} s_{i_2} \cdots s_{i_{m-1}}(\alpha_{i_m}^\vee)\big) \ . \tag{3.2}$$

It is well-known that $a_k \in \mathbf{Z}_+$ for all k.

Define the element $v(\mathbf{i}) = v(\mathbf{i})^\lambda \in \mathcal{A}$ by:

$$v(\mathbf{i}) = F_{i_m}^{a_m} F_{i_{m-1}}^{a_{m-1}} \cdots F_{i_1}^{a_1} \cdot 1 \ . \tag{3.3}$$

The following result refines Proposition 0.6.

Theorem 3.1. *In the above notation, we have $\psi_{\mathbf{i}}(v(\mathbf{i})) = c t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m}$ where $c \in k(q), c \neq 0$.*

Proposition 0.6 follows by taking any λ with $\lambda(\alpha_{i_1}^\vee) = l_{i_1} > 0$ (and $x := c^{-1} v(\mathbf{i})$).

Proof of Theorem 3.1. Let us reformulate our statement in terms of modules over the quantized enveloping algebra $\mathbf{U} = U_q(\mathbf{g})$. The $k(q)$-algebra $\mathbf{U}$ is generated by $F_1, \ldots, F_r$, $E_1, \ldots, E_r$ and the invertible pairwise commuting elements $K_1, \ldots, K_r$ subject to the following relations (see [8]):

$$K_i E_j K_i^{-1} = q^{C_{ij}} E_j, \ K_i F_j K_i^{-1} = q^{-C_{ij}} E_j K_i, \quad E_i F_j - F_j E_i = \delta_{ij} \frac{K - K^{-1}}{v_i - v_i^{-1}} \ ; \tag{3.4}$$

$$\sum_{p+p'=1-a_{ij}} (-1)^p v_i^{pp'} E_i^{[p]} E_j E_i^{[p']} = 0, \quad \sum_{p+p'=1-a_{ij}} (-1)^p v_i^{pp'} F_i^{[p]} F_j F_i^{[p']} = 0 \ . \tag{3.5}$$

The relations (3.5) are *quantum Serre relations*; they hold for all $i \neq j$ where $E_i^{[n]}$ means the same as in (2.2), and $v_i = q^{\frac{C_{ii}}{2}}$.

The between $E_1, \ldots, E_r$ (the same relations between $F_1, \ldots, F_r$):

We identify the subalgebra generated by $E_1, \ldots, E_r$ with $\mathcal{U}$. Note that $\mathbf{U}$ is a $\mathbf{Z}^r$-graded algebra via $\deg(K_i) = \deg(K_i^{-1}) = 0$, $\deg(E_i) = \alpha_i$, $\deg(F_i) = -\alpha_i$ for $i = 1, \ldots, r$. Note also, that each triple (E_i, F_i, K_i) generates the subalgebra in $\mathbf{U}$ isomorphic to $U_{v_i}(sl_2)$.

We will denote by the same symbol E_i the operator $E_i : \mathcal{A} \to \mathcal{A}$ adjoint of the operator of the right multiplication $E \to EE_i$ in $\mathcal{U}$; for every $x \in \mathcal{A}$ the element $E_i \cdot x \in \mathcal{A}$ is defined by the equations $(E_i \cdot x)(E) = x(EE_i)$ for all $E \in \mathcal{U}$. We also define the operator $K_i : \mathcal{A} \to \mathcal{A}$ (depending on λ) by

$$K_i \cdot x = K_{i,\lambda} \cdot x := q^{\lambda(\alpha_i)-(\gamma,\alpha_i)}x \tag{3.6}$$

for all $x \in \mathcal{A}(\gamma)$, $i = 1, \ldots, r$.

The following result is well-known, (for type A_r it can be found e.g, in [3]).

Proposition 3.2. *For every λ as above, the operators F_i defined in (3.1) together with the K_i and E_i, give rise to an action $\mathbf{U} \times \mathcal{A} \to \mathcal{A}$.*

Denote by V_λ the cyclic $\mathbf{U}$-submodule in $\mathcal{A}$ generated by the unit $1 \in \mathcal{A}$. It is well-known (cf. [3], [8]) that V_λ is an integrable simple $\mathbf{U}$-module. Note also that the vector $v = 1 \in V_\lambda$ is a *highest weight vector of weight* λ since $E_i \cdot v = 0$ and $K_i(v) = q^{\lambda(\alpha_i)}v$ for all i.

It is also known (see [8]), that the element $v(\mathbf{i})$ given by (3.3) depends only on w. Such elements are called *extremal vectors* in V_λ. Denote $\mathbf{i}_k := (i_1, \ldots, i_k)$ for $k = 1, \ldots, m$. It is also well-known that for all k we have

$$F_{i_k} \cdot v(\mathbf{i}_k) = 0,\ E_{i_k}^a \cdot v(\mathbf{i}_k) = c_{k;a}F_{i_k}^{a_k-a}v(\mathbf{i}_{k-1}),\ E_{i_k} \cdot v(\mathbf{i}_{k-1}) = 0 \tag{3.7}$$

for some $c_{k;a} \in k(q) \setminus \{0\}$, and $a = 0, 1, \ldots, a_k$ (with the agreement $v(\mathbf{i}_0) := v$).

In view of (2.2), Theorem 3.1 is equivalent to the following.

Proposition 3.3. *There is a unique sequence $b = (b_1, \ldots, b_m)$ such that*

$$E_{i_1}^{b_1} \cdots E_{i_m}^{b_m} \cdot v(\mathbf{i}) = cv \tag{3.8}$$

for some nonzero scalar $c \in k(q)$, namely, $(b_1, \ldots, b_m) = (a_1, \ldots, a_m)$, where $a_1, \ldots, a_m$ are given by (3.2).

Proof of Proposition 3.3. We proceed by induction on m.

Assume that our statement is true for every reduced expression of length $< m$, in particular, for $\mathbf{i}_{m-1}$.

Step 1. Let us prove that the equality (3.8) implies that $b_k = a_k$ for all k with $i_k \neq i_m$.

We will use the following identity in $\mathbf{U}$, which is a straightforward consequence of the relations (3.4):

$$E_{i_1}^{b_1}E_{i_2}^{b_2} \cdots E_{i_m}^{b_m}F_{i_m}^{a_m} = \sum_{b'} F_{i_m}^{|b'|-|b|+a_m}E_{i_1}^{b'_1}E_{i_2}^{b'_2} \cdots E_{i_m}^{b'_m}p_{b'} \tag{3.9}$$

where the sum is over all $b' = (b'_1, \dots, b'_m) \in \mathbf{Z}_+^m$ such that $b'_k = b_k$ if $i_k \neq i_m$, and $b'_k \leq b_k$ if $i_k = i_m$; each $p_{b'}$ is a Laurent polynomial of K_{i_m}, and $|b| = b_1 + \cdots b_m$.

Using (3.9) and the fact that $v(\mathbf{i}) = F_{i_m}^{a_m} \cdot v(\mathbf{i}_{m-1})$, we rewrite (3.8) as follows:

$$cv = E_{i_1}^{b_1} \cdots E_{i_m}^{b_m} F_{i_m}^{a_m} \cdot v(\mathbf{i}_{m-1}) = \sum F_{i_m}^{|b'|-|b|+a_m} E_{i_1}^{b'_1} \cdots E_{i_m}^{b'_m} p_{b'} \cdot v(\mathbf{i}_{m-1}) \tag{3.10}$$

where the sum is over all $(b'_1, \dots, b'_m)$ such that $b'_k = b_k$ whenever $i_k \neq i_m$.

It follows that, for some b', we have

$$F_i^{|b'|-|b|+a_m} E_{i_1}^{b'_1} \cdots E_{i_{m-1}}^{b'_m} p_{b'} \cdot v(\mathbf{i}_{m-1}) = c'v$$

with $c' \in k(q), c' \neq 0$. By (3.7), we have $|b'| - |b| + a_m = 0$ and $b'_m = 0$; also,

$$E_{i_1}^{b'_1} \cdots E_{i_{m-1}}^{b'_{m-1}} \cdot v(\mathbf{i}_{m-1}) = c''v$$

with $c'' \neq 0$. Remembering the inductive assumption, we see that $b'_k = a_k$ for all $k \leq m-1$. Thus, $b_k = b'_k = a_k$ for all $k \leq m-1$ such that $i_k \neq i_m$. This completes **Step 1**.

Step 2. Let us prove that $b_m = a_m$. If $i_k \neq i_m$ for $k = 1, \dots, m-1$ then the equality $b_m = a_m$ follows by comparing degrees. So we can assume that $i_k = i_m$ for some $k < m$. Let $k < m$ be the maximal index such that $i_k = i_m$. Clearly, $k \leq m-2$ since $\mathbf{i}$ is reduced. By **Step 1**, we have $b_{k+1} = a_{k+1},\ b_{k+2} = a_{k+2}, \dots, b_{m-1} = a_{m-1}$. Combining this observation with (3.7), we can rewrite the left hand side of (3.10) as follows.

$$cv = dE_{i_1}^{b_1} \cdots E_{i_{m-1}}^{b_{m-1}} F_{i_m}^{a_m - b_m} \cdot v(\mathbf{i}_{m-1}) = dE_{i_1}^{b_1} \cdots E_{i_k}^{b_k} E_{i_{k+1}}^{a_{k+1}} \cdots E_{i_{m-1}}^{a_{m-1}} F_{i_m}^{a_m - b_m} \cdot v(\mathbf{i}_{m-1})$$

for some $d \in k(q), d \neq 0$.

Then, by the commutativity property $E_{i_l} F_{i_m} = F_{i_m} E_{i_l}$ for $k < l < m$, the previous expression is equal to

$$cv = dE_{i_1}^{b_1} \cdots E_{i_k}^{b_k} F_{i_m}^{a_m - b_m} E_{i_{k+1}}^{a_{k+1}} \cdots E_{i_{m-1}}^{a_{m-1}} \cdot v(\mathbf{i}_{m-1}) = d' E_{i_1}^{b_1} \cdots E_{i_k}^{b_k} F_{i_m}^{a_m - b_m} \cdot v(\mathbf{i}_k) \tag{3.11}$$

(we again used the property (3.7) of the extremal vectors). Since $i_k = i_m$, it follows that $F_{i_m} \cdot v(\mathbf{i}_k) = 0$. Hence, the right hand side of (3.11) is zero unless $a_m - b_m = 0$. This completes **Step 2**.

Step 3. Now we are able to complete the proof. Since $b_m = a_m$, (3.7) and (3.11) imply that

$$c_m E_{i_1}^{b_1} \cdots E_{i_{m-1}}^{b_{m-1}} \cdot v(\mathbf{i}_{m-1}) = cv$$

with some nonzero constants c, c_m. We conclude that $(b_1, \dots, b_{m-1}) = (a_1, \dots, a_{m-1})$ by the inductive assumption. Combining this with **Step 2**, we see that $(b_1, \dots, b_m) = (a_1, \dots, a_m)$.

Proposition 3.3 and Theorem 3.1 are proved. $\triangleleft$

Appendix. Skew-fields of fractions and skew polynomials

Let A be an associative ring with unit without zero-divisors. As in [7], A.2, we say that A satisfies the *right Ore* condition if $aA \cap bA \neq \{0\}$ for any non-zero $a, b \in A$. The set of right fractions $\mathcal{F}(A)$ is defined as the set of all pairs (a, b) with $a, b \in A, b \neq 0$ modulo the following equivalence relation: $(a, b) \sim (c, d)$ if there are $f, g \in A \setminus \{0\}$ such that $af = cg$ and $bf = dg$. The equivalence class of (a, b) in $\mathcal{F}(A)$ is denoted by ab^{-1}. The ring A is naturally embedded into $\mathcal{F}(A)$ via $a \mapsto (a, 1)$. It is well known that if A satisfies the right Ore condition then the addition and multiplication in A extend to $\mathcal{F}(A)$ so that $\mathcal{F}(A)$ becomes a skew-field.

Now we suppose that A is an algebra over a field $\mathbf{k}$ with an increasing filtration $\big(\mathbf{k} = A_0 \subset A_1 \subset \cdots\big)$, where each A_k is a finite dimensional $\mathbf{k}$-vector space, $A_k A_l \subset A_{k+l}$, and $A = \cup A_k$. We say that A has *polynomial growth* if for all $n \geq 0$ we have $\dim A_n \leq p(n)$, where $p(x)$ is a polynomial. For the convenience of the reader, we will present a proof of the following well known lemma (see, e.g., [5]).

Lemma A1. *Any algebra of polynomial growth without zero-divisors satisfies the right Ore condition.*

Proof. Assume, on the contrary, that $aA \cap bA = \{0\}$ for some non-zero $a, b \in A$. Denote $I_n = I \cap A_n$ for any subspace $I \subset A$. Choose some k such that $a, b \in A_k$. Then $(aA)_{n+k} \supset aA_n$ and $(bA)_{n+k} \supset bA_n$, which implies

$$\dim(aA)_{n+k} \geq \dim A_n,\ \dim(bA)_{n+k} \geq \dim A_n\ .$$

On the other hand, since $aA \cap bA = \{0\}$, it follows that

$$\dim A_{n+k} \geq \dim(aA)_{n+k} + \dim(bA)_{n+k} \geq 2 \dim A_n$$

for all n. Iterating this inequality, we see that $\dim A_{mk} \geq 2^m$ for $m \geq 0$. This contradicts the condition that A has polynomial growth. Lemma A1 is proved. $\triangleleft$

Lemma A1 implies that any subalgebra of an algebra A of polynomial growth without zero-divisors also satisfies the right Ore condition.

In particular, consider the $\mathbf{k}$-algebra P of skew polynomials generated by $t_1, \ldots, t_m$ subject to the relations $t_l t_k = q_{kl} t_l t_k$ for $1 \leq k < l \leq m$, where the q_{kl} are some non-zero elements of $\mathbf{k}$. It is easy to see that P has no zero-divisors and has polynomial growth with respect to the filtration $(\mathbf{k} = P_0 \subset P_1 \subset \cdots)$, where P_n is the linear span of all monomials in $t_1, \ldots, t_m$ of degree $\leq n$. We see that every subalgebra $\mathcal{B}$ of P satisfies the right Ore condition. Therefore, $\mathcal{F}(\mathcal{B})$ is a skew subfield of $\mathcal{F}(P)$.

References

[1] A.Berenstein, S.Fomin, A.Zelevinsky, Parametrizations of canonical bases and totally positive matrices. To appear in *Adv. in Math.*

[2] A.Berenstein, A.Zelevinsky, String bases for quantum groups of type A_r, *Advances in Soviet Math.*, **16**, Part 1 (1993), 51–89.

[3] A.Berenstein, A.Zelevinsky, Canonical bases for the quantum group of type A_r and piecewise-linear combinatorics. To appear in *Duke Math. J.*.

[4] L.D.Faddeev, N.Yu.Reshetikhin, L A.Takhtadzhyan, Quantization of Lie groups and Lie algebras. (Russian) *Algebra i Analiz*, **1** (1989), no. 1, 178–206.

[5] K.Iohara, F.Malikov, Rings of skew polynomials and Gel'fand-Kirillov conjecture for quantum groups, *Commun. Math. Phys.* **164** (1994), 217–238.

[6] A.Joseph, Sur une conjecture de Feigin. *C. R. Acad. Sci. Paris Sér.I Math.* **320** (1995), no. 12, 1441–1444.

[7] A.Joseph, Quantum groups and their primitive ideals, Springer-Verlag, Berlin, 1995.

[8] G.Lusztig, Introduction to quantum groups, Birkhauser, Boston, 1993.

[9] G.Lusztig, Problems on canonical bases. Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991), 169–176, *Proc. Sympos. Pure Math.*, **56**, Part 2, Amer. Math. Soc., Providence, RI, 1994.

[10] S.Majid, Algebras and Hopf algebras in braided categories. Advances in Hopf algebras (Chicago, IL, 1992), 55–105, *Lecture Notes in Pure and Appl. Math.*, **158**, Dekker, New York, 1994.

[11] A.Morozov, L.Vinet, Free-field representation of group element for simple quantum group. Preprint ITEP-M3/94, CRM-2202, *hep-th* 9409093.

[12] A.Panov, Skew fields of twisted rational functions and the skew field of rational functions on $\mathrm{GL}_q(n,K)$. *St. Petersburg Math. J*, **7** (1996), no. 1.